PLANE AND SPHERICAL TRIGONOMETRY

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PREFACE.

Many American text-books on trigonometry treat the solution of triangles quite fully; English text-books elaborate analytical trigonometry; but no book available seems to meet both needs adequately. To do that is the first aim of the present work, in the preparation of which nearly everything has been worked out and tested by the authors in their classes.

The work entered upon, other features demanded attention. For some unaccountable reason nearly all books, in proving the formulæ for functions of $\alpha \pm \beta$, treat the same line as both positive and negative, thus vitiating the proof; and proofs given for acute angles are (without further discussion) supposed to apply to all angles, or it is suggested that the student can draw other figures and show that the formulæ hold in all cases. As a matter of fact the average student cannot show anything of the kind; and if he could, the proof would still apply only to combinations of conditions the same as those in the figures actually drawn. These difficulties are avoided by so wording the proofs that the language applies to figures involving any angles, and to avoid drawing the indefinite number of figures necessary fully to establish the formulæ geometrically, the general case is proved algebraically (see page 58).

Inverse functions are introduced early, and used constantly. Wherever computations are introduced they are made by means of logarithms. The average student, using logarithms for a short time and only at the end of the subject, straightway forgets what manner of things they are. It is hoped, by dint of much practice, extended over as long a time as possible, to give the student a command of logarithms that will stay. The fundamental formulæ of trigonometry must be memorized. There is no substitute for this. For this purpose oral work is introduced, and there are frequent lists of review problems involving all principles and formulæ previously developed. These lists serve the

further purpose of throwing the student on his own resources, and compelling him to find in the problem itself, and not in any model solution, the key to its solution, thus developing power, instead of ability to imitate. To the same end, in the solution of triangles, divisions and subdivisions into cases are abandoned, and the student is thrown on his own judgment to determine which of the three possible sets of formulæ will lead to the solutions with the data given. Long experience justifies this as clearer and simpler. The use of checks is insisted upon in all computations.

For the usual course in plane trigonometry Chapters I–VII, omitting Arts. 26, 27, contain enough. Articles marked * (as Art. * 26) may be omitted unless the teacher finds time for them without neglecting the rest of the work. Classes that can accomplish more will find a most interesting field opened in the other chapters. More problems are provided than any student is expected to solve, in order that different selections may be assigned to different students, or to classes in different years. *Do not assign work too fast. Make sure the student has memorized and can use each preceding formula, before taking up new ones.*

No complete acknowledgment of help received could here be made. The authors are under obligation to many for general hints, and to several who, after going over the proof with care, have given valuable suggestions. The standard works of Levett and Davison, Hobson, Henrici and Treutlein, and others have been freely consulted, and while many of the problems have been prepared by the authors in their class-room work, they have not hesitated to take, from such standard collections as writers generally have drawn upon, any problems that seemed better adapted than others to the work. Quality has not been knowingly sacrificed to originality. Corrections and suggestions will be gladly received at any time.

E. A. L., Ypsilanti.

E. C. G., Ann Arbor.

October, 1900.

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Chapter X. Spherical Trigonometry.

Chapter XI. Solution of Spherical Triangles.

PLANE TRIGONOMETRY.

CHAPTER I.

ANGLES — MEASUREMENT OF ANGLES.

1. Angles. It is difficult, if not impossible, to define an angle. This difficulty may be avoided by telling how it is formed. *If a line revolve about one of its points, an angle is generated*, the magnitude of the angle depending on the amount of the rotation.

Thus, if one side of the angle θ, as OR, be originally in the position OX, and be revolved about the point O to the position in the figure, the angle XOR is generated. OX is called the *initial* line, and any position of OR the *terminal* line of the angle formed. The angle θ is considered *positive if generated by a counter-clockwise rotation of* OR, and hence *negative if generated by a clockwise rotation.* The magnitude of θ depends on the amount of rotation of OR, and since the amount of such rotation may be unlimited, there is no limit to the possible magnitude of angles, for, evidently, the revolving line may reach the position OR by rotation through an acute angle θ, and, likewise, by rotation through once, twice, $\cdots$, n times $360°$, plus the acute angle θ. So that XOR may mean the acute angle θ, $\theta + 360°$, $\theta + 720°$, $\cdots$, $\theta + n \cdot 360°$.

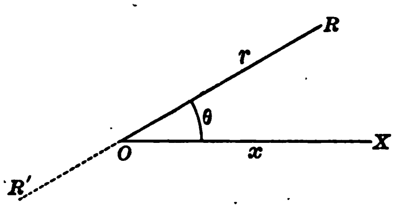


FIG. 1.

In reading an angle, read first the initial line, then the terminal line. Thus in the figure the acute angle XOR, or xr, is a positive angle, and ROX, or rx, an equal negative angle.

Ex. 1. Show that if the initial lines for $\frac{1}{2}$, $\frac{9}{2}$, $\frac{25}{2}$, $-\frac{7}{2}$, right angles are the same, the terminal lines may coincide.

2. Name four other angles having the same initial and terminal lines as $\frac{1}{3}$ of a right angle; as $\frac{5}{4}$ of a right angle; as $\frac{3}{2}$ of a right angle.

2. Rectangular axes. Any plane surface may be divided by two perpendicular straight lines XX' and YY' into four portions, or *quadrants*.

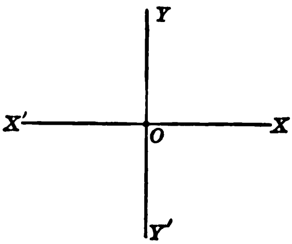


FIG. 2.

XX' is known as the *x-axis*, YY' as the *y-axis*, and the two together are called *axes of reference*. Their intersection O is the *origin*, and the four portions of the plane surface, XOY, YOX', $X'OY'$, $Y'OX$, are called respectively the *first*, *second*, *third*, and *fourth quadrants*. The position of any point in the plane is determined when we know its *distances* and *directions* from the axes.

3. Any direction may be considered positive. Then the opposite direction must be negative. Thus, if AB represents any positive line, BA is an equal negative line.

Mathematicians usually consider *lines measured in the same direction as OX or OY* (Fig. 2) *as positive. Then lines measured in the same direction as OX' or OY' must be negative.*

The distance of any point from the y-axis is called the *abscissa*, its distance from the x-axis the *ordinate*, of that point; the two together are the *coördinates* of the point, usually denoted by the letters x and y respectively, and written (x, y).

When taken with their proper signs, the coördinates define completely the position of the point. Thus, if the point P is $+a$ units from YY', and $+b$ units from XX', any convenient unit of length being chosen, the position of P is known. For we have only to measure a distance ON equal to a units along OX, and then from N measure a distance b units parallel to OY, and we arrive at the position of the point P, (a, b). In like manner we may locate P', $(-a, b)$, in the second quadrant, P'', $(-a, -b)$, in the third quadrant, and P''', $(a, -b)$, in the fourth quadrant.

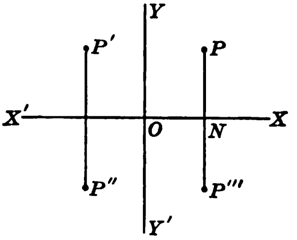


FIG. 3.

Ex. Locate $(2, -2)$; $(0, 0)$; $(-8, -7)$; $(0, 5)$; $(-2, 0)$; $(2, 2)$; (m, n).

4. If OX is the initial line, θ is said to be an *angle of the first, second, third,* or *fourth quadrant,* according as its terminal line is in the first, second, third, or fourth quadrant. It is clear that as OR rotates its *quality* is in no way affected, and hence it is *in all positions considered positive*, and its extension through O, OR', negative.

The student should notice that the initial line may take any position and revolve in either direction. While it is customary to consider the counter-clockwise rotation as forming a positive angle, yet the conditions of a figure may be such that a positive angle may be generated by a clockwise rotation. Thus the angle XOR in each figure may be traced as a positive angle by revolving the initial line OX to the position OR. No confusion can result if the fact is clear that when an angle is read XOR, OX is considered a positive line revolving to the position OR. OX' and OR' then are negative lines in whatever directions drawn. These conceptions are mere matters of agreement, and the agreement may be determined in a particular case by the conditions of the problem quite as well as by such general agreements of mathematicians as those referred to in Arts. 3 and 4 above.

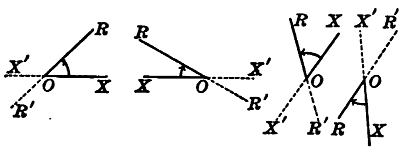


FIG. 4.

5. Measurement. All measurements are made in terms of some fixed standard adopted as a unit. This unit must

be of the same kind as the quantity measured. Thus, length is measured in terms of a unit length, surface in terms of a unit surface, weight in terms of a unit weight, value in terms of a unit value, an angle in terms of a unit angle.

The measure of a given quantity is the number of times it contains the unit selected.

Thus the area of a given surface in square feet is the number of times it contains the unit surface 1 sq. ft.; the length of a road in miles, the number of times it contains the unit length 1 mi.; the weight of a cargo of iron ore in tons, the number of times it contains the unit weight 1 ton; the value of an estate, the number of times it contains the unit value $1.

The same quantity may have different measures, according to the unit chosen. So the measure of 80 acres, when the unit surface is 1 acre, is 80, when the unit surface is 1 sq. rd., is 12,800, when the unit surface is 1 sq. yd., is 387,200. What is its measure in square feet?

6. The essentials of a good unit of measure are:

1. *That it be invariable, i.e.* under all conditions bearing the same ratio to equal magnitudes.

2. *That it be convenient* for practical or theoretical purposes.

3. *That it be of the same kind as the quantity measured.*

7. Two systems of measuring angles are in use, the *sexagesimal* and the *circular*.

The *sexagesimal* system is used in most practical applications. The right angle, the unit of measure in geometry, though it is invariable, as a measure is too large for convenience. Accordingly it is divided into 90 equal parts, called *degrees*. The degree is divided into 60 *minutes*, and the minute into 60 *seconds*. Degrees, minutes, seconds, are indicated by the marks $^\circ$ $'$ $''$, as $36^\circ\ 20'\ 15''$.

The division of a right angle into hundredths, with subdivisions into hundredths, would be more convenient. The French have proposed such

a *centesimal* system, dividing the right angle into 100 grades, the grade into 100 minutes, and the minute into 100 seconds, marked ${}^{g}\ {}^{\backprime}\ {}^{\backprime\backprime}$, as $50^{g}\ 70^{\backprime}\ 28^{\backprime\backprime}$. The great labor involved in changing mathematical tables, instruments, and records of observation to the new system has prevented its adoption.

8. The *circular* system is important in theoretical considerations. It is based on the fact that for a given angle the ratio of the length of its arc to the length of the radius of that arc is constant, *i.e.* for a fixed angle the ratio *arc : radius* is the same no matter what the length of the radius. In the figure, for the angle θ,

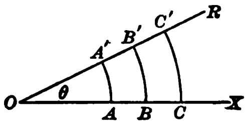


FIG. 5.

$$\frac{OA}{AA'} = \frac{OB}{BB'} = \frac{OC}{CC'} = \cdots.$$

That this ratio of arc to radius for a fixed angle is constant follows from the established geometrical principles:

1. The circumference of any circle is 2π times its radius.
2. Angles at the centre are in the same ratio as their arcs.

The Radian. It follows that an angle whose arc is equal in length to the radius is a constant angle for all circles, since in four right angles, or the perigon, there are always 2π such angles. *This constant angle, whose arc is equal in length to the radius, is taken as the unit angle of circular measure, and is called the radian.* From the definition we have

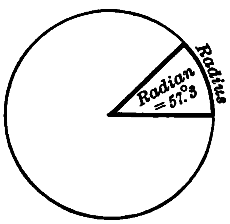


FIG. 6.

$$4 \text{ right angles} = 360° = 2\pi \text{ radians},$$

$$2 \text{ right angles} = 180° = \pi \text{ radians},$$

$$1 \text{ right angle} = 90° = \frac{\pi}{2} \text{ radians}.$$

π is a numerical quantity, 3.14159+, and not an angle. When we speak of 180° as π, 90° as $\frac{\pi}{2}$, etc., we always mean π *radians*, $\frac{\pi}{2}$ *radians*, etc.

9. To change from one system of measurement to the other we use the relation,

$$2\pi \text{ radians} = 360°.$$

$$\therefore 1 \text{ radian} = \frac{180°}{\pi} = 57°.2958-;$$

i.e. the radian is 57°.3, *approximately.*

Ex. 1. Express in radians 75° 30′.

$$75°\ 30' = 75°.5;\ 1 \text{ radian} = 57°.3.$$

$$\therefore 75°\ 30' = \frac{75.5}{57.3} = 1.317 \text{ radians.}$$

2. Express in degree measure 3.6 radians.

$$1 \text{ radian} = 57°.3.$$

$$\therefore 3.6 \text{ radians} = 3.6 \times 57°.3 = 206°\ 16'\ 48''.$$

EXAMPLES.

1. Construct, approximately, the following angles: 50°, – 20°, 90°, 179°, – 135°, 400°, – 380°, 1140°, $\frac{\pi}{4}$ radians, $\frac{\pi}{3}$ radians, $-\frac{\pi}{6}$ radians, 3π radians, $-\frac{3\pi}{4}$ radians, $\frac{12\pi}{5}$ radians. Of which quadrant is each angle?

2. What is the measure of:

   (*a*) $\frac{4}{5}$ of a right angle, when 30° is the unit of measure?
   (*b*) an acre, when a square whose side is 10 rds. is the unit?
   (*c*) *m* miles, when *y* yards is the unit?

3. What is the unit of measure, when the measure of $2\frac{1}{2}$ miles is 50?

4. The Michigan Central R.R. is 535 miles long, and the Ann Arbor R.R. is 292 miles long. Express the length of the first in terms of the second as a unit.

5. What will be the measure of the radian when the right angle is taken for the unit? Of the right angle when the radian is the unit?

6. In which quadrant is 45°? 10°? – 60°? 145°? 1145°? – 725°? Express each in right angles; in radians.

7. Express in sexagesimal measure

$$\frac{\pi}{3}, \frac{\pi}{12}, 1,\ 6.28, \frac{1}{\pi}, \frac{7\pi}{3}, -\frac{4\pi}{3}, \text{ radians.}$$

8. Express in each system an interior angle of a regular hexagon; an exterior angle.

9. Find the distance in miles between two places on the earth's equator which are $11° 15'$ apart. (The earth's radius is about 3963 miles.)

10. Find the length of an arc which subtends an angle of 4 radians at the centre of a circle of radius 12 ft. 3 in.

11. An arc 15 yds. long contains 3 radians. Find the radius of the circle.

12. Show that the hour and minute hands of a watch turn through angles of $30'$ and $6°$ respectively per minute; also find in degrees and in radians the angle turned through by the minute hand in 3 hrs. 20 mins.

13. Find the number of seconds in an arc of 1 mile on the equator; also the length in miles of an arc of $1'$ (1 knot).

14. Find to three decimal places the radius of a circle in which the arc of $71° 36' 3''.6$ is 15 in. long.

15. Find the ratio of $\frac{\pi}{6}$ to $5°$.

16. What is the shortest distance measured on the earth's surface from the equator to Ann Arbor, latitude $+42° 16' 48''$?

17. The difference of two angles is $10°$, and the circular measure of their sum is 2. Find the circular measure of each angle.

18. A water wheel of radius 6 ft. makes 30 revolutions per minute. Find the number of miles per hour travelled by a point on the rim.

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

10. Trigonometry, as the word indicates, was originally concerned with the measurement of triangles. It now includes the analytical treatment of certain *functions of angles*, as well as the solution of triangles by means of certain relations between the functions of the angles of those triangles.

11. Function. If one quantity depends upon another for its value, the first is called a *function* of the second. It always follows that the second quantity is also a function of the first; and, in general, functions are so related that if one is constant the other is constant, and if either varies in value, the other varies. This relation may be extended to any number of mutually dependent quantities.

Illustration. If a train moves at a rate of 30 miles per hour, the distance travelled is a function of the rate and time, the time is a function of the rate and distance, and the rate is a function of the time and distance.

Again, the circumference of a circle is a function of the radius, and the radius of the circumference, for so long as either is constant the other is constant, and if either changes in value, the other changes, since circumference and radius are connected by the relation $C = 2\pi R$.

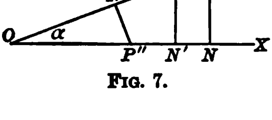


FIG. 7.

Once more, in the right triangle NOP, the ratio of any two sides is a function of the angle α, because all the right triangles of which α is one angle are similar, *i.e.* the ratio

of two corresponding sides is constant so long as α is constant, and varies if α varies.

Thus, the ratios

$$\frac{NP}{OP}=\frac{N'P'}{OP'}=\frac{N''P''}{OP''}$$

and $$\frac{ON}{NP}=\frac{ON'}{N'P'}=\frac{ON''}{N''P''}, \text{ etc.,}$$

depend on α for their values, *i.e.* are functions of α.

12. The trigonometric functions. In trigonometry six *functions of angles* are usually employed, called the *trigonometric functions.*

By definition *these functions are the six ratios between the sides of the triangle of reference of the given angle. The triangle of reference is formed by drawing from some point in the initial line, or the initial line produced, a perpendicular to that line meeting the terminal line of the angle.*

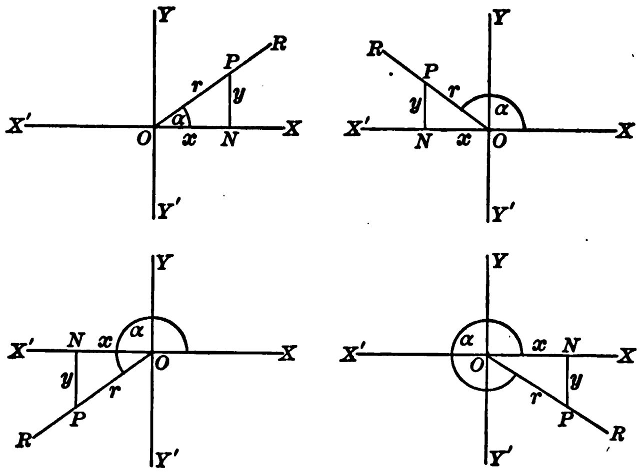


FIG. 8.

Let α be an angle of any quadrant. Each triangle of reference of α, NOP, is formed by drawing a perpendicular to OX, or OX produced, meeting the terminal line OR in P.

If α is greater than 360°, its triangle of reference would not differ from one of the above triangles.

It is perhaps worthy of notice that the *triangle of reference* might be defined to be the triangle formed by drawing a perpendicular to either side of the angle, or that side produced, meeting the other side or the other side produced. In the figure, NOP is in all cases the triangle of reference of α. The principles of the following pages are the same no matter which of the triangles is considered the triangle of reference. It will, however, be as well, and perhaps clearer, to use the triangle defined under Fig. 8, and we shall always draw the triangle as there described.

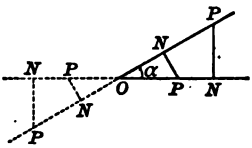


FIG. 9.

13. The trigonometric functions of α (Fig. 8) are called the *sine*, *cosine*, *tangent*, *cotangent*, *secant*, and *cosecant* of α. These are abbreviated in writing to sin α, cos α, tan α, cot α, sec α, csc α, and are defined as follows:

$$\sin\alpha = \frac{\text{perp.}}{\text{hyp.}} = \frac{y}{r}, \text{ whence } y = r\,\sin\alpha;$$

$$\cos\alpha = \frac{\text{base}}{\text{hyp.}} = \frac{x}{r}, \text{ whence } x = r\,\cos\alpha;$$

$$\tan\alpha = \frac{\text{perp.}}{\text{base}} = \frac{y}{x}, \text{ whence } y = x\,\tan\alpha;$$

$$\cot\alpha = \frac{\text{base}}{\text{perp.}} = \frac{x}{y}, \text{ whence } x = y\,\cot\alpha;$$

$$\sec\alpha = \frac{\text{hyp.}}{\text{base}} = \frac{r}{x}, \text{ whence } r = x\,\sec\alpha;$$

$$\csc\alpha = \frac{\text{hyp.}}{\text{perp.}} = \frac{r}{y}, \text{ whence } r = y\,\csc\alpha.$$

$1 - \cos\alpha$ and $1 - \sin\alpha$, called *versed-sine* α and *coversed-sine* α, respectively, are sometimes used.

Ex. 1. Write the trigonometric functions of β, NPO (Fig. 8), and compare with those of α above.

The meaning of the prefix *co* in cosine, cotangent, and cosecant appears from the relations of Ex. 1. For the *sine of an angle* equals the *cosine*, *i.e.* the *complement-sine*, of the *complement of that angle;* the *tangent*

of an angle equals the *cotangent of its complementary angle,* and the *secant of an angle* equals the *cosecant of its complementary angle.*

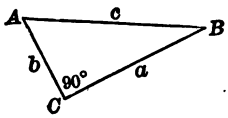


FIG. 10.

2. Express each side of triangle ABC in terms of another side, and some function of an angle in all possible ways, as $a = b \tan A$, etc.

14. Constancy of the trigonometric functions. It is important to notice why these ratios are *functions of the angle, i.e.* are the same for equal angles and different for unequal angles. This is shown by the principles of similar triangles.

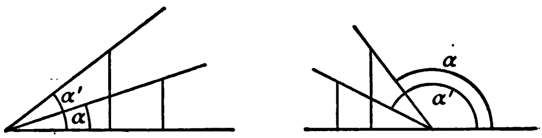


FIG. 11.

In each figure show that in all possible triangles of reference for α the ratios are the same, but in the triangles of reference for α and α', respectively, the ratios are different.

The student must notice that $\sin\alpha$ is a *single symbol.* It is the *name of a number,* or *fraction,* belonging to the angle α; and if it be at any time convenient, we may denote $\sin\alpha$ by a *single letter,* such as o, or x. Also, $\sin^2\alpha$ is an abbreviation for $(\sin\alpha)^2$, *i.e.* for $(\sin\alpha)\times(\sin\alpha)$. Such abbreviations are used because they are convenient. Lock, *Elementary Trigonometry.*

15. Fundamental relations. From the definitions of Art. 13 the following reciprocal relations are apparent:

$$\sin\alpha = \frac{1}{\csc\alpha}, \qquad \csc\alpha = \frac{1}{\sin\alpha},$$

$$\cos\alpha = \frac{1}{\sec\alpha}, \qquad \sec\alpha = \frac{1}{\cos\alpha},$$

$$\tan\alpha = \frac{1}{\cot\alpha}, \qquad \cot\alpha = \frac{1}{\tan\alpha}.$$

Also from the definitions,

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha}, \qquad \cot\alpha = \frac{\cos\alpha}{\sin\alpha}.$$

From the right triangle *NOP*, page 9,

$$y^2 + x^2 = r^2;$$

whence (1) $$\frac{y^2}{r^2} + \frac{x^2}{r^2} = 1,$$

(2) $$\frac{y^2}{x^2} + 1 = \frac{r^2}{x^2},$$

(3) $$1 + \frac{x^2}{y^2} = \frac{r^2}{y^2}.$$

From (1) $\sin^2 \alpha + \cos^2 \alpha = 1$; $\sin \alpha = \sqrt{1 - \cos^2 \alpha}$; $\cos \alpha = ?$

(2) $\tan^2 \alpha + 1 = \sec^2 \alpha$; $\tan \alpha = \sqrt{\sec^2 \alpha - 1}$; $\sec \alpha = ?$

(3) $1 + \cot^2 \alpha = \csc^2 \alpha$; $\cot \alpha = \sqrt{\csc^2 \alpha - 1}$; $\csc \alpha = ?$

The foregoing definitions and fundamental relations are of the highest importance, and must be mastered at once. The student of trigonometry is helpless without perfect familiarity with them.

These relations are true for all values of α, positive or negative, but the signs of the functions are not in all cases positive, as appears from the fact that in the triangles of reference in Fig. 8 x and y are sometimes negative. The equations $\sin \alpha = \pm\sqrt{1 - \cos^2 \alpha}$, $\tan \alpha = \pm\sqrt{\sec^2 \alpha - 1}$, $\cot \alpha = \pm\sqrt{\csc^2 \alpha - 1}$, have the double sign $\pm$. Which sign is to be used in a given case depends on the quadrant in which α lies.

16. The relations of Art. 15 enable us to express any function in terms of any other, or when one function is given, to find all the others.

Ex. 1. To express the other functions in terms of tangent:

$$\sin \alpha = \frac{1}{\csc \alpha} = \frac{1}{\sqrt{1 + \cot^2 \alpha}} = \frac{\tan \alpha}{\sqrt{1 + \tan^2 \alpha}}; \quad \cot \alpha = \frac{1}{\tan \alpha};$$

$$\cos \alpha = \frac{1}{\sec \alpha} = \frac{1}{\sqrt{1 + \tan^2 \alpha}}; \quad \sec \alpha = \sqrt{1 + \tan^2 \alpha};$$

$$\tan \alpha = \tan \alpha; \quad \csc \alpha = \frac{\sqrt{1 + \tan^2 \alpha}}{\tan \alpha}.$$

In like manner determine the relations to complete the following table:

	$\sin\alpha$	$\cos\alpha$	$\tan\alpha$	$\cot\alpha$	$\sec\alpha$	$\csc\alpha$
$\sin\alpha$			$\dfrac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$			
$\cos\alpha$			$\dfrac{1}{\sqrt{1+\tan^2\alpha}}$			
$\tan\alpha$			$\tan\alpha$			
$\cot\alpha$			$\dfrac{1}{\tan\alpha}$			
$\sec\alpha$			$\sqrt{1+\tan^2\alpha}$			
$\csc\alpha$			$\dfrac{\sqrt{1+\tan^2\alpha}}{\tan\alpha}$			

2. Given $\sin\alpha = \frac{3}{4}$; find the other functions.

$$\cos\alpha = \sqrt{1-\tfrac{9}{16}} = \tfrac{1}{4}\sqrt{7};\quad \tan\alpha = \frac{\frac{3}{4}}{\frac{1}{4}\sqrt{7}} = \frac{3}{\sqrt{7}} = \tfrac{3}{7}\sqrt{7};$$

$$\cot\alpha = \frac{1}{\frac{3}{7}\sqrt{7}} = \tfrac{1}{3}\sqrt{7};\quad \sec\alpha = \frac{1}{\frac{1}{4}\sqrt{7}} = \frac{4}{\sqrt{7}} = \tfrac{4}{7}\sqrt{7};\quad \csc\alpha = \frac{1}{\frac{3}{4}} = \frac{4}{3}.$$

3. Given $\tan\phi + \cot\phi = 2$; find $\sin\phi$.

$$\tan\phi + \frac{1}{\tan\phi} = 2,\quad \tan^2\phi - 2\tan\phi + 1 = 0,\quad \tan\phi = 1.$$

$$\therefore\ \sin\phi = \frac{\tan\phi}{\sqrt{1+\tan^2\phi}} = \tfrac{1}{2}\sqrt{2}.$$

Or, expressing in terms of sine directly, $\dfrac{\sin\phi}{\cos\phi} + \dfrac{\cos\phi}{\sin\phi} = 2$,

$$\sin^2\phi + \cos^2\phi = 2\sin\phi\cos\phi,\quad \sin^2\phi - 2\sin\phi\cos\phi + \cos^2\phi = 0;$$

whence $\sin\phi - \cos\phi = 0,\ \sin\phi = \cos\phi.\ \ \therefore\ \sin\phi = \tfrac{1}{2}\sqrt{2}.$

4. Prove $\sec^4 x - \sec^2 x = \tan^2 x + \tan^4 x$.

$$\sec^4 x - \sec^2 x = \sec^2 x(\sec^2 x - 1) = (1+\tan^2 x)\tan^2 x = \tan^2 x + \tan^4 x.$$

5. Prove $\sin^6 y + \cos^6 y = 1 - 3\sin^2 y\cos^2 y$.

$$\begin{aligned}\sin^6 y + \cos^6 y &= (\sin^2 y + \cos^2 y)(\sin^4 y - \sin^2 y\cos^2 y + \cos^4 y)\\ &= (\sin^2 y + \cos^2 y)^2 - 3\sin^2 y\cos^2 y = 1 - 3\sin^2 y\cos^2 y.\end{aligned}$$

6. Prove $\dfrac{\tan z}{1-\cot z}+\dfrac{\cot z}{1-\tan z}=\sec z\csc z+1.$

$$\frac{\tan z}{1-\cot z}+\frac{\cot z}{1-\tan z}=\frac{\dfrac{\sin z}{\cos z}}{1-\dfrac{\cos z}{\sin z}}+\frac{\dfrac{\cos z}{\sin z}}{1-\dfrac{\sin z}{\cos z}}$$

$$=\frac{\sin^2 z}{\cos z(\sin z-\cos z)}+\frac{\cos^2 z}{\sin z(\cos z-\sin z)}$$

$$=\frac{\sin^3 z-\cos^3 z}{\sin z\cos z(\sin z-\cos z)}=\frac{\sin^2 z+\sin z\cos z+\cos^2 z}{\sin z\cos z}$$

$$=\frac{1+\sin z\cos z}{\sin z\cos z}=\frac{1}{\sin z\cos z}+1=\sec z\csc z+1.$$

In solving problems like 3, 4, 5, and 6 above, it is usually safe, if no other step suggests itself, to express all other functions of one member in terms of sine and cosine. The resulting expression may then be reduced by the principles of algebra to the expression in the other member of the equation. For further suggestions as to the solution of trigonometric equations and identities see page 66.

EXAMPLES.

1. Find the values of all the functions of α, if $\sin\alpha=\frac{2}{3}$; if $\tan\alpha=\frac{3}{4}$; if $\sec\alpha=2$; if $\cos\alpha=\frac{1}{3}\sqrt{3}$; if $\cot\alpha=\frac{1}{7}$; if $\csc\alpha=\sqrt{2}$.

2. Compute the functions of each acute angle in the right triangles whose sides are: (1) 3, 4, 5; (2) 8, 15, 17; (3) 480, 31, 481; (4) a, b, c; (5) $\dfrac{2xy}{x-y}$, $\dfrac{x^2+y^2}{x-y}$, $x+y$.

3. If $\cos\alpha=\frac{8}{17}$, find the value of $\dfrac{\sin\alpha+\tan\alpha}{\cos\alpha-\cot\alpha}$.

4. If $2\cos\alpha=2-\sin\alpha$, find $\tan\alpha$.

5. If $\sec^2\alpha\csc^2\alpha-4=0$, find $\cot\alpha$.

6. Solve for $\sin\beta$ in $\quad 13\sin\beta+5\cos^2\beta=11$.

Prove

7. $\sin^4\phi-\cos^4\phi=1-2\cos^2\phi$.

8. $(\sin\alpha+\cos\alpha)(\sin\alpha-\cos\alpha)=2\sin^2\alpha-1$.

9. $(\sec\alpha+\tan\alpha)(\sec\alpha-\tan\alpha)=1$.

10. $\cos^2\beta(\sec^2\beta-2\sin^2\beta)=\cos^4\beta+\sin^4\beta$.

11. $\tan v+\sec v=\dfrac{\cos v}{1-\sin v}$.

12. $\dfrac{\sin w}{1-\cos w}=\dfrac{1+\cos w}{\sin w}$.

13. $(\sec\theta+1)(1-\cos\theta)=\tan^2\theta\cos\theta$.

14. $\sin^4 t - \sin^2 t = \cos^4 t - \cos^2 t.$

15. $\dfrac{\sin\beta}{1-\sin\beta} + \dfrac{1+\sin\beta}{\sin\beta} = \sec^2\beta\,(\csc\beta + 1).$

16. $(\tan A + \cot A)^2 = \sec^2 A \csc^2 A.$

17. $\sec^2 x - \sin^2 x = \tan^2 x + \cos^2 x.$

In the triangle ABC, right angled at C,

18. Given $\cos A = \frac{8}{17}$, $BC = 45$, find $\tan B$, and AB.

19. If $\cos A = \dfrac{m^2 - n^2}{m^2 + n^2}$, and $AB = m^2 + n^2$, find AC and BC.

20. If $AC = m + n$, $BC = m - n$, find $\sin A$, $\cos B$.

21. In examples 18, 19, 20, above, prove $\sin^2 A + \cos^2 A = 1$; $1 + \tan^2 A = \sec^2 A$.

17. Functions of certain angles. The trigonometric functions are numerical quantities which may be determined for any angle. In general these values are taken from tables prepared for the purpose, but the principles already studied enable us to calculate the functions of the following angles.

18. Functions of 0°. If α be a very small angle, the value of y is very small, and decreases as α diminishes. Clearly, when α approaches 0° as a limit, y likewise approaches 0, and x approaches r, so that when $\alpha = 0°$,

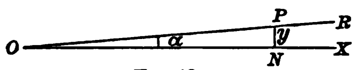


FIG. 12.

$$y = 0, \text{ and } x = r.$$

$$\therefore\ \textit{sin}\ 0° = \frac{y}{r} = 0, \qquad \textit{cot}\ 0° = \frac{1}{\tan 0°} = \infty,$$

$$\textit{cos}\ 0° = \frac{x}{r} = 1, \qquad \textit{sec}\ 0° = \frac{1}{\cos 0°} = 1,$$

$$\textit{tan}\ 0° = \frac{y}{x} = 0, \qquad \textit{csc}\ 0° = \frac{1}{\sin 0°} = \infty.$$

In the figure of Art. 18, by diminishing α it is clear that we can make y as small as we please, and by making α small enough, we can make the *value of y less than any assignable quantity, however small, so that $\sin\alpha$ approaches as a limit* 0. This is what we mean when we say $\sin 0° = 0$. In like manner, it is evident that, by sufficiently diminishing α we can make *$\cot\alpha$ greater than any assignable* quantity. This we express by saying *cot* $0° = \infty$.

19. Functions of 30°. Let NOP be the triangle of reference for an angle of 30°. Make triangle $NOP' = NOP$. Then POP' is an equilateral triangle (why?), and ON bisects PP'. Hence

$$PP' = r = 2y.$$

Also $x = \sqrt{r^2 - y^2} = \sqrt{3y^2} = y\sqrt{3}$.

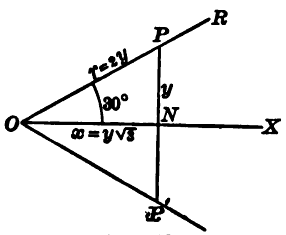


FIG. 13.

$$\therefore \ \sin 30° = \frac{y}{r} = \frac{y}{2y} = \frac{1}{2}, \qquad \csc 30° = 2,$$

$$\cos 30° = \frac{x}{r} = \frac{y\sqrt{3}}{2y} = \tfrac{1}{2}\sqrt{3}, \qquad \sec 30° = \tfrac{2}{3}\sqrt{3},$$

$$\tan 30° = \frac{y}{x} = \frac{y}{y\sqrt{3}} = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3}, \qquad \cot 30° = \sqrt{3}.$$

20. Functions of 45°. Let NOP be the triangle of reference. If angle $NOP = 45°$, $OPN = 45°$.

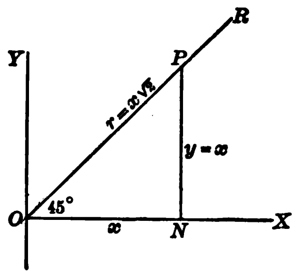


FIG. 14.

$$\therefore \ y = x, \text{ and } r = \sqrt{x^2 + y^2} = \sqrt{2x^2} = x\sqrt{2}.$$

Then
$$\sin 45° = \frac{y}{r} = \frac{x}{x\sqrt{2}} = \tfrac{1}{2}\sqrt{2},$$

$$\cos 45° = \frac{x}{r} = \frac{x}{x\sqrt{2}} = \tfrac{1}{2}\sqrt{2},$$

$$\tan 45° = \frac{y}{x} = \frac{x}{x} = 1.$$

Find $\cot 45°$, $\sec 45°$, $\csc 45°$.

21. Functions of 60°. The functions of 60° may be computed by means of the figure, or they may be written from the functions of the complement, or 30°. Let the student in both ways show that

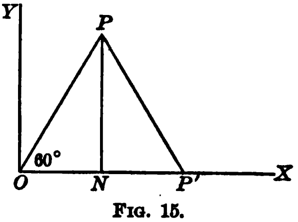


FIG. 15.

$$\sin 60° = \tfrac{1}{2}\sqrt{3}, \quad \cos 60° = \tfrac{1}{2},$$
$$\tan 60° = \sqrt{3}.$$

Compute also the other functions of 60°.

22. Functions of 90°. If α be an angle very near 90°, the value of x is very small, and decreases as α increases toward 90°. Clearly when α approaches 90° as a limit, x approaches 0, and y approaches r, so that when

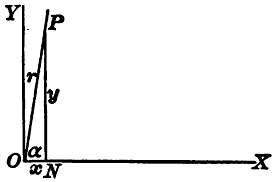


FIG. 16.

$$\alpha = 90°, \quad x = 0, \quad y = r.$$

$$\therefore \sin 90° = 1, \ \cos 90° = 0, \ \tan 90° = \infty.$$

Compute the other functions. Also find the functions of 90° from those of its complement, 0°.

23. It is of great convenience to the student to remember the functions of these angles. *They are easily found by recalling the relative values of the sides of the triangles of reference for the respective angles*, or the values of the other functions may readily be computed by means of the fundamental relations, if the values of the sine and cosine are remembered, as follows:

α	0°	30°	45°	60°	90°
sine	$\frac{1}{2}\sqrt{0}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{4}$
cosine	$\frac{1}{2}\sqrt{4}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}\sqrt{2}$	$\frac{1}{2}\sqrt{1}$	$\frac{1}{2}\sqrt{0}$

ORAL WORK.

1. Which is greater, $\sin 45°$ or $\frac{1}{2}\sin 90°$? $\sin 60°$ or $2\sin 30°$?

2. From the functions of 60°, find those of 30°; from the functions of 90°, those of 0°. Why are the functions of 45° equal to the co-functions of 45°?

3. Given $\sin A = \frac{1}{3}$, find $\cos A$; $\tan A$.

4. Show that $\sin B \csc B = 1$; $\cos C \sec C = 1$; $\cot x \tan x = 1$.

5. Show that $\sec^2\theta - \tan^2\theta = \csc^2\theta - \cot^2\theta = \sin^2\theta + \cos^2\theta$.

6. Show that $\tan 30° \tan 60° = \cot 60° \cot 30° = \tan 45°$.

7. Show that $\tan 60° \sin^2 45° = \cos 30° \sin 90°$.

8. Show that $\cos\alpha \tan\alpha = \sin\alpha$; $\sin\beta \cot\beta = \cos\beta$.

9. Show that $\dfrac{1-\tan^2 30°}{1+\tan^2 30°} = \cos 60° = \frac{1}{2}\cos 0°$.

10. Show that $(\tan y + \cot y)\sin y \cos y = 1$.

EXAMPLES.

1. Show that $\sin 30° \cos 60° + \cos 30° \sin 60° = \sin 90°$.

2. Show that $\cos 60° \cos 30° + \sin 60° \sin 30° = \cos 30°$.

3. Show that $\sin 45° \cos 0° - \cos 45° \sin 0° = \cos 45°$.

4. Show that $\cos^2 45° - \sin^2 45° = \cos 90°$.

5. Show that $\dfrac{\tan 45° + \tan 0°}{1 - \tan 45° \tan 0°} = \tan 45°$.

If $A = 60°$, verify

6. $\sin\dfrac{A}{2} = \sqrt{\dfrac{1-\cos A}{2}}$.

7. $\tan\dfrac{A}{2} = \sqrt{\dfrac{1-\cos A}{1+\cos A}}$.

8. $\cos A = 2\cos^2\dfrac{A}{2} - 1 = 1 - 2\sin^2\dfrac{A}{2}$.

If $\alpha = 0°$, $\beta = 30°$, $\gamma = 45°$, $\delta = 60°$, $\epsilon = 90°$, find the values of

9. $\sin\beta + \cos\delta$.

10. $\cos\beta + \tan\delta$.

11. $\sin\beta\cos\delta + \cos\beta\sin\delta - \sin\epsilon$.

12. $(\sin\beta + \sin\epsilon)(\cos\alpha + \cos\delta) - 4\sin\alpha(\cos\gamma + \sin\epsilon)$.

24. Variations in the trigonometric functions.

Signs. Thus far no account has been taken of the *signs of the functions.* By the definitions it appears that these depend on the signs of x, y, and r. Now r is always positive, and from the figures it is seen that x is positive in the first

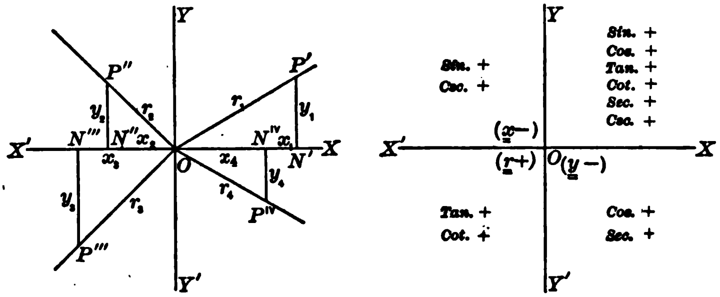


FIG. 17.

and fourth quadrants, and y is positive in the first and second. Hence

For an angle in the *first quadrant* all functions are *positive*, since x, y, r are *positive*.

In the *second quadrant* x alone is *negative*, so that those functions whose ratios involve x, viz. *cosine*, *tangent*, *cotangent*, *secant*, are *negative;* the others, *sine* and *cosecant*, are *positive*.

In the *third quadrant* x and y are both *negative*, so that those functions involving r, viz. *sine*, *cosine*, *secant*, *cosecant*, are *negative;* the others, *tangent* and *cotangent*, are *positive*.

In the *fourth quadrant* y is *negative*, so that *sine*, *tangent*, *cotangent*, *cosecant* are *negative*, and *cosine* and *secant*, *positive*.

Values. In the triangle of reference of any angle, the hypotenuse r is never less than x or y. Then if r be taken of any fixed length, as the angle varies, the base and perpendicular of the triangle of reference may each vary in length from 0 to r. Hence the ratios $\frac{x}{r}$ and $\frac{y}{r}$ can never be greater than 1, nor if x and y are negative, less than -1; and $\frac{r}{x}$, $\frac{r}{y}$

cannot have values between $+1$ and -1. But the ratios $\frac{y}{x}$ and $\frac{x}{y}$ may vary without limit, *i.e.* from $+\infty$ to $-\infty$. Therefore the possible values of the functions of an angle are:

sine and cosine between $+1$ and -1,

i.e. *sine and cosine cannot be numerically greater than* 1;

tangent and cotangent between $+\infty$ and $-\infty$,

i.e. *tangent and cotangent may have any real value;*

secant and cosecant between $+\infty$ and $+1$, and -1 and $-\infty$,

i.e. *secant and cosecant may have any real values, except values between* $+1$ and -1.

These limits are indicated in the following figures. The student should carefully verify.

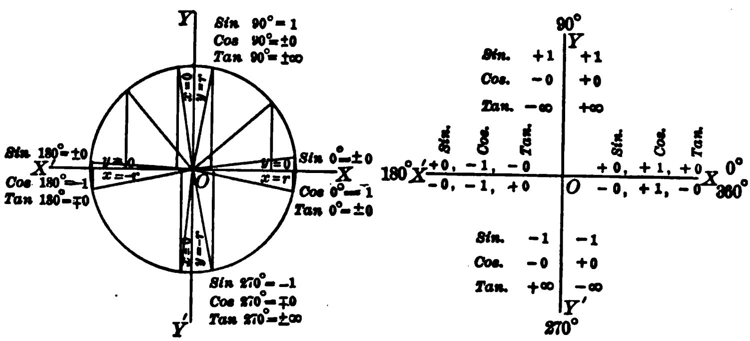


FIG. 18.

25. In tracing the changes in the values of the functions as α changes from 0° to 360°, consider the revolving line r as of fixed length. Then x and y may have any length between 0 and r.

Sine. At 0°, $\sin\alpha = \frac{y}{r} = \frac{0}{r} = 0$. As α increases through the first quadrant, y increases from 0 to r, whence $\frac{y}{r}$ increases from 0 to 1. In passing to 180° $\sin\alpha$ decreases from 1 to 0,

since y decreases from r to 0. As α passes through 180°, y changes sign, and in the third quadrant decreases to negative r, so that $\sin \alpha$ decreases from 0 to -1. In the fourth quadrant y increases from negative r to 0, and hence $\sin \alpha$ increases from -1 to 0.

Cosine depends on changing values of x. Show that, as α increases from 0° to 360°, $\cos \alpha$ varies in the four quadrants as follows: 1 to 0, 0 to -1, -1 to 0, 0 to 1.

Tangent depends on changing values of both y and x.

$$\text{At } 0°,\ y = 0,\ x = r, \qquad \text{at } 180°,\ y = 0,\ x = -r,$$

$$\text{at } 90°,\ x = 0,\ y = r, \qquad \text{at } 270°,\ x = 0,\ y = -r.$$

Hence $\tan 0° = \frac{y}{x} = \frac{0}{r} = 0$. As α passes to 90°, y increases to r, and x decreases to 0, so that $\tan \alpha$ increases from 0 to ∞. As α passes through 90°, x changes sign, so that $\tan \alpha$ changes from positive to negative by passing through ∞. In the second quadrant x decreases to negative r, y to 0, and $\tan \alpha$ passes from $-\infty$ to 0. As α passes through 180°, $\tan \alpha$ changes from minus to plus by passing through 0, because at 180° y changes to minus. In the third quadrant $\tan \alpha$ passes from 0 to ∞, changing sign at 270° by passing through ∞, because at 270° x changes to plus. In the fourth quadrant $\tan \alpha$ passes from $-\infty$ to 0.

Cotangent. In like manner show that $\cot \alpha$ passes through the values ∞ to 0, 0 to $-\infty$, ∞ to 0, 0 to $-\infty$, as α passes from 0° to 360°.

Secant depends on x for its value. Noting the change in x as under cosine, we see that secant passes from 1 to ∞, $-\infty$ to -1, -1 to $-\infty$, ∞ to 1.

Cosecant passes through the values ∞ to 1, 1 to ∞, $-\infty$ to -1, -1 to $-\infty$.

The student should trace the changes in each function fully, as has been done for sine and tangent, giving the reasons at each step.

α	0° to 90°	90° to 180°	180° to 270°	270° to 360°
sin	0 to 1	1 to 0	-0 to -1	-1 to -0
cos	1 to 0	-0 to -1	-1 to -0	0 to 1
tan	0 to ∞	$-\infty$ to -0	0 to ∞	$-\infty$ to -0
cot	∞ to 0	-0 to $-\infty$	∞ to 0	-0 to $-\infty$
sec	1 to ∞	$-\infty$ to -1	-1 to $-\infty$	∞ to 1
csc	∞ to 1	1 to ∞	$-\infty$ to -1	-1 to $-\infty$

*** 26. Graphic representation of functions.** These variations are clearly brought out by graphic representations of the functions. Two cases will be considered: I, when α is a constant angle; II, when α is a variable angle.

I. *When α is a constant angle.*

The trigonometric functions are ratios, pure numbers. By so choosing the triangle of reference that the denominator of the ratio is a side of unit length, the side forming the numerator of that ratio will be a geometrical representation of the value of that function, *e.g.* if in Fig. 19 $r = 1$, then $\sin \alpha = \frac{y}{r} = \frac{y}{1} = y$. This may be done by making α a central angle in a circle of radius 1, and drawing triangles of reference as follows:

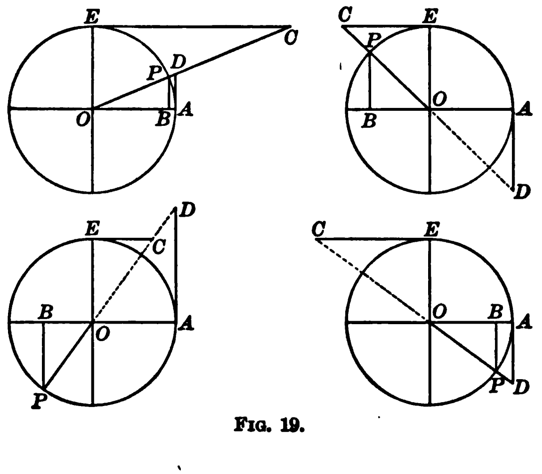


FIG. 19.

In all the figures $AOP = \alpha$, and

$$\sin\alpha = \frac{BP}{OP} = \frac{BP}{1} = BP,$$

$$\cos\alpha = \frac{OB}{OP} = \frac{OB}{1} = OB,$$

$$\tan\alpha = \frac{BP}{OB} = \frac{AD}{OA} = \frac{AD}{1} = AD,$$

$$\cot\alpha = \frac{OA}{AD} = \frac{EC}{OE} = \frac{EC}{1} = EC,$$

$$\sec\alpha = \frac{OP}{OB} = \frac{OD}{OA} = \frac{OD}{1} = OD,$$

$$\csc\alpha = \frac{OP}{BP} = \frac{OC}{OE} = \frac{OC}{1} = OC.$$

It appears then that, by taking a radius 1,

sine is represented by the perpendicular to the initial line, drawn from that line to the terminus of the arc subtending the given angle;

cosine is represented by the line from the vertex of the angle to the foot of the sine;

tangent is represented by the geometrical tangent drawn from the origin of the arc to the terminal line, produced if necessary;

cotangent is represented by the geometrical tangent drawn from a point 90° from the origin of the arc to the terminal line, produced if necessary;

secant is represented by the terminal line, or the terminal line produced, from the origin to its intersection with the tangent line;

cosecant is represented by the terminal line, or the terminal line produced, from the origin to its intersection with the cotangent line.

These lines are not the functions, but in triangles drawn as explained their lengths are equal to the numerical values of the functions, and in this sense the lines may be said to *represent* the functions. It will be noticed also that their directions indicate the *signs* of the functions. Let the student by means of these representations verify the results of Arts. 24 and 25.

II. *When α is a variable angle.*

Take XX' and YY' as axes of reference, and let angle units be measured along the x-axis, and values of the functions parallel to the y-axis, as in Art. 3. We may write corresponding values of the angle and the functions thus:

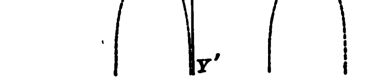
$$\alpha = 0°,\ 30°,\ 45°,\ 60°,\ 90°,\ 120°,\ 135°,\ 150°,\ 180°,\ 210°,\ 225°,$$

$$\sin\alpha = 0,\ \tfrac{1}{2},\ \tfrac{1}{2}\sqrt{2},\ \tfrac{1}{2}\sqrt{3},\ 1,\ \tfrac{1}{2}\sqrt{3},\ \tfrac{1}{2}\sqrt{2},\ \tfrac{1}{2},\ 0,\ -\tfrac{1}{2},\ -\tfrac{1}{2}\sqrt{2},$$

$$\alpha = 240°,\ 270°,\ 300°,\ 315°,\ 330°,\ 360°,\ -30°,\ -45°,\ -60°,\ -90°,\ \text{etc.},$$

$$\sin\alpha = -\tfrac{1}{2}\sqrt{3},\ -1,\ -\tfrac{1}{2}\sqrt{3},\ -\tfrac{1}{2}\sqrt{2},\ -\tfrac{1}{2},\ 0,\ -\tfrac{1}{2},\ -\tfrac{1}{2}\sqrt{2},\ -\tfrac{1}{2}\sqrt{3},\ -1,\ \text{etc.}$$

These values will be sufficient to determine the form of the curve representing the function. By taking angles between those above, and computing the values of the function, as given in mathematical tables, the form of the curve can be determined to any required degree of accuracy. Reducing the above fractions to decimals, it will be convenient to make the y-units large in comparison with the x-units. In the figure one x-unit represents 15°, and one y-unit 0.25.



Curves of Sine and Cosecant.

Sine ————

Cosecant ------------

FIG. 20.

Measuring the angle values along the x-axis, and from these points of division measuring the corresponding values of $\sin\alpha$ parallel to the y-axis, as in Art. 3, we have, approximately,

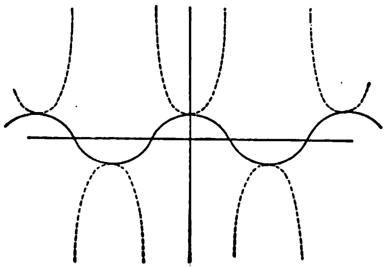
$$OX_1 = 30° = 2 \text{ units}, \qquad OX_2 = 45° = 3 \text{ units},$$
$$X_1Y_1 = \tfrac{1}{2} = 2 \text{ units}, \qquad X_2Y_2 = 0.71 = 2.84 \text{ units},$$

$$OX_3 = 60° = 4 \text{ units, etc.},$$
$$X_3Y_3 = 0.86 = 3.44 \text{ units, etc.}$$

We have now only to draw through the points Y_1, Y_2, Y_3, etc., thus determined, a continuous curve, and we have the *sine-curve* or *sinusoid.*

The dotted curve in the figure is the *cosecant curve.* Let the student compute values, as above, and draw the curve.

In like manner draw the *cosine* and *secant* curves, as follows:



Curves of Cosine and Secant.
Cosine ———
Secant ----------

FIG. 21.

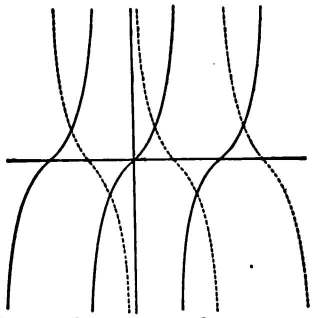
Tangent curve. Compute values for the angle α and for $\tan \alpha$, as before:

$$\alpha = 0°, 30°, 45°, 60°, 90°, 120°, 135°, 150°, 180°, 210°, 225°, 240°, 270°,$$
$$\tan \alpha = 0, \tfrac{1}{3}\sqrt{3}, 1, \sqrt{3}, \pm\infty, -\sqrt{3}, -1, -\tfrac{1}{3}\sqrt{3}, 0, \tfrac{1}{3}\sqrt{3}, 1, \sqrt{3}, \pm\infty,$$

$$\alpha = -30°, -45°, -60°, -90°, \text{etc.},$$
$$\tan \alpha = -\tfrac{1}{3}\sqrt{3}, -1, -\sqrt{3}, \pm\infty, \text{etc.}$$

Then lay off the values of α and of $\tan \alpha$ along the x, and parallel to the y-axis, respectively. It will be noted that,

as α approaches 90°, tan α increases to ∞, and when α passes 90°, tan α is negative. Hence the value is measured parallel



Curves of Tangent and Cotangent.
Tangent ———
Cotangent---------

FIG. 22.

to the y-axis downward, thus giving a discontinuous curve, as in the figure.

* **27.** The following principles are illustrated by the curves:

1. The sine and cosine are continuous for varying values of the angle, and lie within the limits $+1$ and -1. Sine changes sign as the angle passes through 180°, 360°, ⋯, $n\,180°$, while cosine changes sign as the angle passes through 90°, 270°, ⋯, $(2n+1)\,90°$. Tangent and cotangent are discontinuous, the one as the angle approaches 90°, 270°, ⋯, $(2n+1)\,90°$, the other as the angle approaches 180°, 360°, ⋯, $n\,180°$, and each changes sign as the angle passes through these values. The limiting values of tangent and cotangent are $+\infty$ and $-\infty$.

2. A line parallel to the y-axis cuts any of the curves in but one point, showing that for any value of α there is but one value of any function of α. But a line parallel to the x-axis cuts any of the curves in an indefinite number of points, if at all, showing that for any value of the function there are an indefinite number of values, if any, of α.

3. The curves afford an excellent illustration of the variations in sign and value of the functions, as α varies from 0 to 360°, as discussed in Art. 25. Let the student trace these changes.

4. From the curves it is evident that the functions are *periodic*, *i.e.* each increase of the angle through 360° in the case of the sine and cosine, or through 180° in the case of the tangent and cotangent, produces a portion of the curve like that produced by the first variation of the angle within those limits.

5. The difference in rapidity of change of the functions at different values of α is important, and reference will be made to this in computations of triangles. (See Art. 64, Case III.) A glance at the curves shows that sine is changing in value rapidly at 0°, 180°, etc., while near 90°, 270°, etc., the rate of change is slow. But cosine has a slow rate of change at 0°, 180°, etc., and a rapid rate at 90°, 270°, etc. Tangent and cotangent change rapidly throughout.

Ex. Let the student discuss secant and cosecant curves.

ORAL WORK.

1. Express in radians 180°, 120°, 45°; in degrees, $\frac{11}{7}$ radians, 2π, $\frac{2}{3}\pi$, $\frac{3}{2}\pi$.

2. If $\frac{1}{3}$ of a right angle be the unit, what is the measure of $\frac{1}{2}$ of a right angle? of 90°? of 135°?

3. Which is greater, $\cos 30°$ or $\frac{1}{2}\cos 60°$? $\tan\frac{\pi}{6}$ or $\cot\frac{\pi}{3}$? $\sin\frac{\pi}{4}$ or $\cos\frac{\pi}{4}$?

4. Express $\sin\alpha$ in terms of $\sec\alpha$; of $\tan\alpha$; $\tan\alpha$ in terms of $\cos\alpha$; of $\sec\alpha$.

5. Given $\sin\alpha = \frac{3}{5}$, find $\tan\alpha$. If $\tan\alpha = 1$, find $\sin\alpha$, $\csc\alpha$, $\cot\alpha$; also $\tan 2\alpha$, $\sin 2\alpha$, $\cos 2\alpha$.

6. If $\cos\alpha = \frac{1}{2}$, find $\sin\frac{\alpha}{2}$, $\tan\frac{\alpha}{2}$.

7. In what quadrant is angle t, if both $\sin t$ and $\cos t$ are minus? if $\sin t$ is plus and $\cos t$ minus? if $\tan t$ and $\cot t$ are both minus? if $\sin t$ and $\csc t$ are of the same sign? Why?

8. Of the numbers 3, $\frac{4}{3}$, -5, $-\frac{1}{3}$, a, $-b$, ∞, 0, which may be a value of $\sin p$? of $\sec p$? of $\tan p$? Why?

EXAMPLES.

1. If $\sin 26^\circ 40' = 0.44880$, find, correct to 0.00001, the cosine and tangent.

2. If $\tan\alpha = \sqrt{3}$, and $\cot\beta = \frac{1}{3}\sqrt{3}$, find $\sin\alpha\cos\beta - \cos\alpha\sin\beta$.

3. Evaluate $\dfrac{\sin 30^\circ \cot 30^\circ - \cos 60^\circ \tan 60^\circ}{\sin 90^\circ \cos 0^\circ}$.

Prove the identities:

4. $\tan A(1 - \cot^2 A) + \cot A(1 - \tan^2 A) = 0$.

5. $(\sin A + \sec A)^2 + (\cos A + \csc A)^2 = (1 + \sec A \csc A)^2$.

6. $\sin^2 x \cos x \csc x - \cos^3 x \csc x \sin^2 x + \cos^4 x \sec x \sin x = \sin^3 x \cos x + \cos^3 x \sin x$.

7. $\tan^2 w + \cot^2 w = \sec^2 w \csc^2 w - 2$.

8. $\sec^2 v + \cos^2 v = 2 + \tan^2 v \sin^2 v$.

9. $\cos^2 t + 1 = 2\cos^3 t \sec t + \sin^2 t$.

10. $\csc^2 t - \sec^2 t = \cos^2 t \csc^2 t - \sin^2 t \sec^2 t$.

11. The sine of an angle is $\dfrac{m^2 - n^2}{m^2 + n^2}$; find the other functions.

12. If $\tan A + \sin A = m$, $\tan A - \sin A = n$, prove $m^2 - n^2 = 4\sqrt{mn}$.

Solve for one function of the angle involved the equations:

13. $\sin\theta + 2\cos\theta = 1$.

14. $\dfrac{\cos\alpha}{\tan\alpha} = \dfrac{3}{2}$.

15. $\sqrt{3}\csc^2\theta = 4\cot\theta$.

16. $2\sin^2 x + \cos x - 1 = 0$.

17. $\sec^2 x - 7\tan x - 9 = 0$.

18. $3\csc y + 10\cot y - 35 = 0$.

19. $\sin^2 v - \frac{7}{3}\cos v - 1 = 0$.

20. $a\sec^2 w + b\tan w + c - a = 0$.

21. If $\dfrac{\sin A}{\sin B} = \sqrt{2}$, $\dfrac{\tan A}{\tan B} = \sqrt{3}$, find A and B.

22. Find to five decimal places the arc which subtends the angle of 1° at the centre of a circle whose radius is 4000 miles.

23. If $\csc A = \frac{2}{3}\sqrt{3}$, find the other functions, when A lies between $\dfrac{\pi}{2}$ and π.

24. In each of two triangles the angles are in G. P. The least angle of one of them is three times the least angle of the other, and the sum of the greatest angles is 240°. Find the circular measure of each of the angles.

CHAPTER III.

FUNCTIONS OF ANY ANGLE—INVERSE FUNCTIONS.

28. By an examination of the figure of Art. 24 it is seen that all the fundamental relations between the functions hold true for any value of α. The table of Art. 16 expresses the functions of α, whatever be its magnitude, in terms of each of the other functions of that angle if the $\pm$ sign be prefixed to the radicals.

The definitions of the trigonometric functions (Art. 12) apply to angles of any size and sign, but it is always possible to express the functions of any angle in terms of the functions of a *positive acute* angle.

The functions of any angle θ, greater than 360°, are the same as those of $\theta \pm n \cdot 360°$, since θ and $\theta \pm n \cdot 360°$ have the same triangle of reference. Thus the functions of 390°, or of 750°, are the same as the functions of $390° - 360°$, or of $750° - 2 \cdot 360°$, *i.e.* of 30°, as is at once seen by drawing a figure. So also the functions of $-315°$, or of $-675°$ are the same as those of $-315° + 360°$, or of $-675° + 2 \cdot 360°$, *i.e.* of 45°.

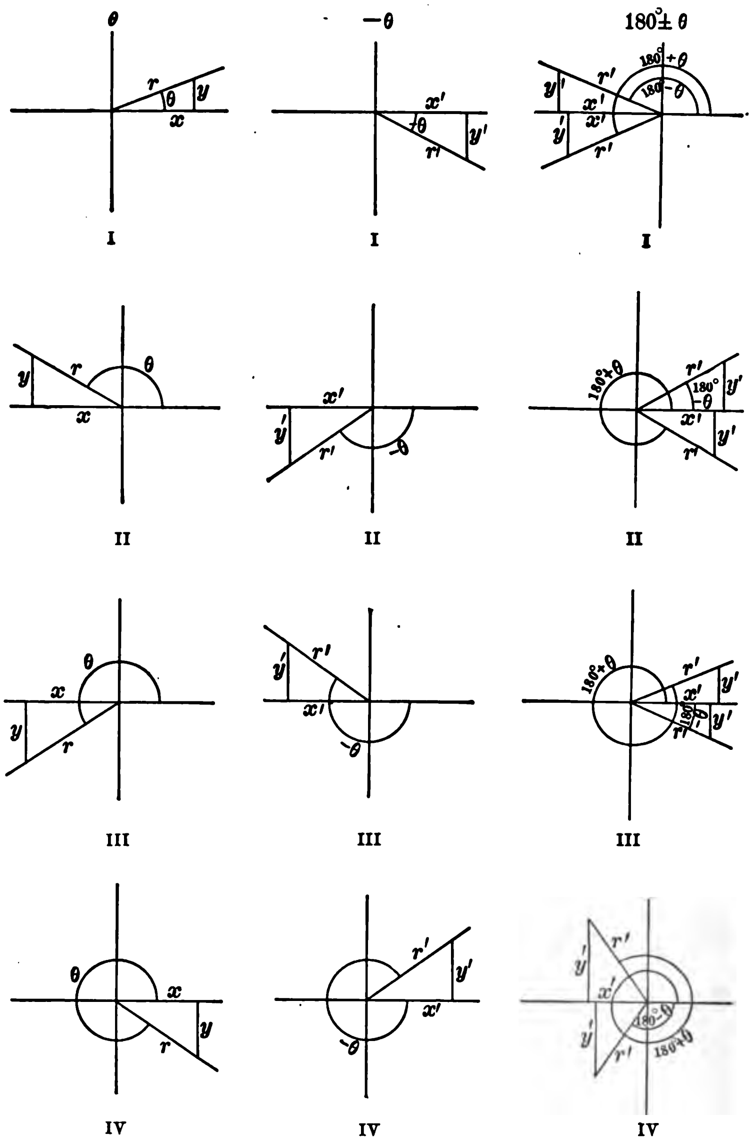
For functions of angles less than 360° the relations of this chapter are important.

29. *To find the relations of the functions of* $-\theta$, $90° \pm \theta$, $180° \pm \theta$, *and* $270° \pm \theta$ *to the functions of* θ, θ *being any angle.*

Four sets of figures are drawn, I for θ an acute angle, II for θ obtuse, III for θ an angle of the third quadrant, and IV for θ an angle of the fourth quadrant.

In every case generate the angles forming the compound angles separately, *i.e.* turn the revolving line first through

(a) (b) (c)


θ
−θ
180° ± θ
r
θ
y
x
x′
−θ
y′
r′
y′
r′
180° + θ
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x′
y′
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I
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180° + θ
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III
θ
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y
r
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r′
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x′
180° − θ
y′
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180° + θ
IV
IV
IV

Fig. 23.

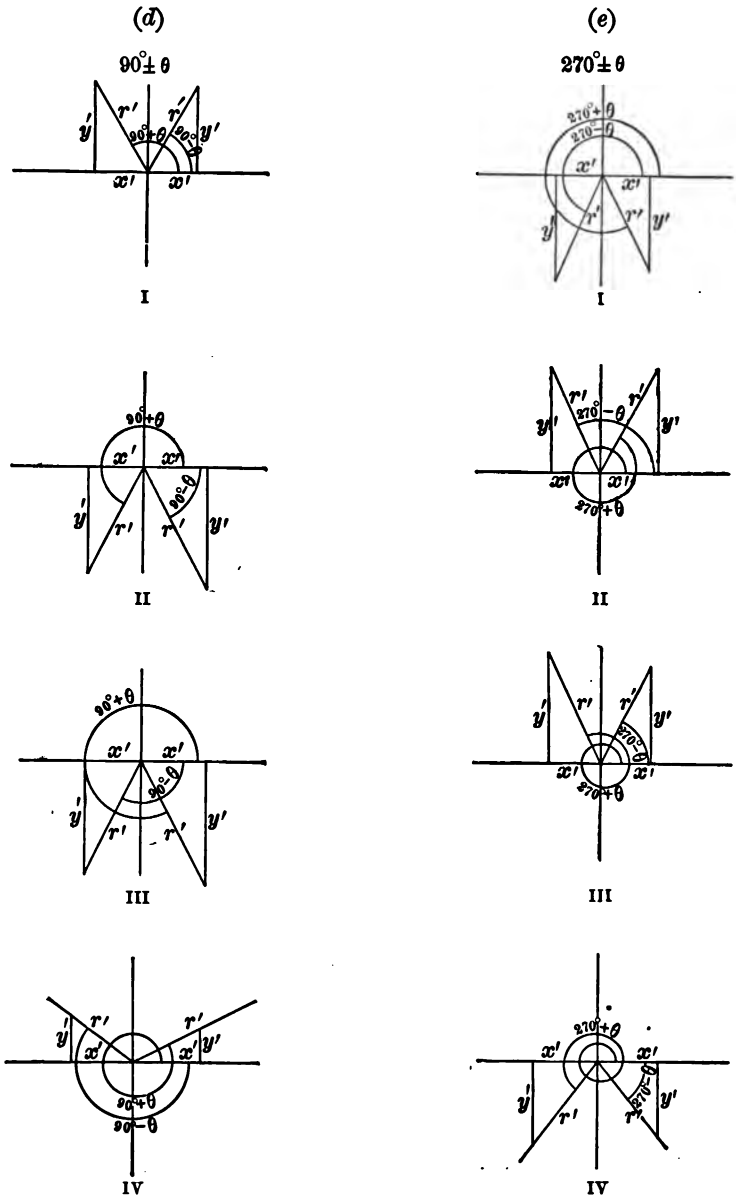


FIG. 23.

$0°$, $90°$, $180°$, or $270°$, and then from this position through θ, or $-\theta$, as the case may be. Form the triangles of reference for (*a*) the angle θ, (*b*) $-\theta$, (*c*) $180° \pm \theta$, (*d*) $90° \pm \theta$, (*e*) $270° \pm \theta$.

The triangles of reference (*a*), (*b*), (*c*), (*d*), and (*e*), in each of the four sets of figures, I, II, III, IV, are similar, being mutually equiangular, since all have a right angle and one acute angle equal each to each. Hence the sides x, y, r of the triangles (*a*) are homologous to x', y', r' of the corresponding triangles (*b*) and (*c*), but to y', x', r', of the corresponding triangles (*d*) and (*e*). For the sides x of triangle (*a*) and x' of the triangles (*b*) and (*c*) are opposite equal angles, and hence are homologous, but the sides y' are opposite this same angle in triangles (*d*) and (*e*), and therefore sides y' of (*d*) and (*e*) are homologous to x of (*a*).

Attending to the signs of x and x', y and y' in the similar triangles (*a*) and (*b*),

$$\sin(-\theta) = \frac{y'}{r'} = -\frac{y}{r} = -\sin\theta,$$

$$\cos(-\theta) = \frac{x'}{r'} = \frac{x}{r} = \cos\theta,$$

$$\tan(-\theta) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\theta.$$

Also in the similar triangles (*a*) and (*c*),

$$\sin(180° - \theta) = \frac{y'}{r'} = \frac{y}{r} = \sin\theta,$$

$$\cos(180° - \theta) = \frac{x'}{r'} = -\frac{x}{r} = -\cos\theta,$$

$$\tan(180° - \theta) = \frac{y'}{x'} = -\frac{y}{x} = -\tan\theta.$$

In like manner show that

$$\sin(180° + \theta) = -\sin\theta,$$

$$\cos(180° + \theta) = -\cos\theta,$$

$$\tan(180° + \theta) = \tan\theta.$$

Again, in the similar triangles (*a*) and (*d*),

$$\sin(90^\circ+\theta)=\frac{y'}{r'}=\frac{x}{r}\quad=\cos\theta,$$

$$\cos(90^\circ+\theta)=\frac{x'}{r'}=-\frac{y}{r}=-\sin\theta,$$

$$\tan(90^\circ+\theta)=\frac{y'}{x'}=-\frac{x}{y}=-\cot\theta.$$

Show that

$$\sin(90^\circ-\theta)=\cos\theta,$$

$$\cos(90^\circ-\theta)=\sin\theta,$$

$$\tan(90^\circ-\theta)=\cot\theta.$$

Finally, from the similar triangles (*a*) and (*e*), show that

$$\sin(270^\circ\pm\theta)=-\cos\theta,$$

$$\cos(270^\circ\pm\theta)=\pm\sin\theta,$$

$$\tan(270^\circ\pm\theta)=\mp\cot\theta.$$

From the reciprocal relations the student can at once write the corresponding relations for secant, cosecant, and cotangent.

30. Since in each of the four cases x', y' of triangles (*b*) and (*c*) are homologous to x, y of triangle (*a*), while x', y' of the triangles (*d*) and (*e*) are homologous to y, x of triangle (*a*), we may express the relations of the last article thus:

The functions of $\begin{cases}\pm\theta\\180^\circ\pm\theta\end{cases}$ *correspond to the same functions of* θ, *while those of* $\begin{cases}90^\circ\pm\theta\\270^\circ\pm\theta\end{cases}$ *correspond to the co-functions of* θ, *due attention being paid to the signs.*

The student can readily determine the sign in any given case, whether θ be acute or obtuse, by considering in what quadrant the compound angle, $90^\circ\pm\theta$, $180^\circ\pm\theta$, etc., would

lie if θ were an acute angle, and prefixing to the corresponding functions of θ the signs of the respective functions for an angle in that quadrant. Thus $90° + \theta$, if θ be acute, is an angle of the second quadrant, so that sine and cosecant are plus, the other functions minus. It will be seen that $\sin(90° + \theta) = +\cos\theta$, $\cos(90° + \theta) = -\sin\theta$, etc., and this will be true whatever be the magnitude of θ. It will assist in fixing in the memory these important relations to notice that when in the compound angle θ is measured from the y-axis, as in $90° \pm \theta$, $270° \pm \theta$, the functions of one angle correspond to the co-functions of the other, but when in the compound angle θ is measured from the x-axis, as in $\pm\theta$, $180° \pm \theta$, then the functions of one angle correspond to the same functions of the other.

These relations, as has been noted in Art. 28, can be extended to angles greater than 360°, and it may be stated generally that

$$\text{function } \theta = \pm \text{ function } (2n \cdot 90° \pm \theta),$$

$$\text{function } \theta = \pm \text{ co-function } [(2n + 1)\,90° \pm \theta].$$

Computation tables contain angles less than 90° only. The chief utility of the above relations will be the reduction of functions of angles greater than 90° to functions of acute angles. Thus, to find $\tan 130° 20'$, look in the tables for $\cot 40° 20'$, or for $\tan 49° 40'$. Why?

Ex. 1. What angles less than 360° have the same numerical cosine as 20°?

$$\cos 20° = -\cos(180° \pm 20°) = \cos(360° - 20°).$$

$\therefore$ 200°, 160°, 340° have the same cosine numerically as 20°.

2. Find the functions of 135°; of 210°.

$$\sin 135° = \sin(90° + 45°) = \cos 45° = \tfrac{1}{2}\sqrt{2},$$

$$\cos 135° = \cos(180° - 45°) = -\cos 45° = -\tfrac{1}{2}\sqrt{2}, \text{ etc.}$$

$$\sin 210° = \sin(180° + 30°) = -\sin 30° = -\tfrac{1}{2}.$$

Let the student give the other functions for each angle.

ORAL WORK.

1. Determine the sine and tangent of each of the following angles: $30°$, $120°$, $-30°$, $-60°$, $\frac{2}{3}\pi$, $2\frac{1}{3}\pi$, $-135°$, $-\pi$.

2. Which is the greater, $\sin 30°$ or $\sin(-30°)$? $\tan 135°$ or $\tan 45°$? $\cos 60°$ or $\cos(-60°)$? $\sin 22° 30'$ or $\cos 67° 30'$?

3. What positive angle has the same tangent as $\frac{\pi}{3}$? the same sine as $50°$?

4. If $\tan \theta = -1$, find $\sin \theta$.

5. Find $\sin 510°$, $\cos(-60°)$, $\tan 150°$.

6. Reduce in two ways to functions of a positive acute angle, $\cos 122°$ $\tan 140° 30'$, $\sin(-60°)$.

7. Find all positive values of x, less than $360°$, satisfying the following equations: $\cos x = \cos 45°$, $\sin 2x = \sin 10°$, $\tan 3x = \tan 60°$, $\sin x = \sin 30°$, $\tan x = \tan 135°$.

8. What angles are determined when (*a*) sine and cosine are $+$? (*b*) cotangent and sine are $-$? (*c*) sine $+$ and cosine $-$? (*d*) cosine $-$ and cotangent $+$?

INVERSE FUNCTIONS.

31. That a is the sine of an angle θ may be expressed in two ways, viz., $\sin \theta = a$, or, inversely, $\theta = \sin^{-1} a$, the latter being read, *θ equals an angle whose sine is a*, or, more briefly, *θ is the anti-sine of a.*

The notation $\sin^{-1} a$, $\cos^{-1} a$, $\tan^{-1} a$, etc., is not a fortunate one, but is so generally accepted that a change is not probable. The symbol may have been suggested from the fact that if $ax = b$, then $x = a^{-1} b$, whence, by analogy, if $\sin \theta = a$, $\theta = \sin^{-1} a$. But the likeness is an analogy only, for there is no similarity in *meaning*. $\text{Sin}^{-1} a$ *is an angle* θ, where $\sin \theta = a$, and is entirely different from $(\sin a)^{-1} = \frac{1}{\sin a}$. In Europe the symbols arc sin a, arc cos a, etc., are employed.

32. Principal value. We have found that in $\sin \theta = a$, for any value of θ, a can have but one value; but in $\theta = \sin^{-1} a$, for any value of a there are an indefinite number of values of θ (Art. 27, 2).

Thus, when $\sin \theta = a$, if $a = \frac{1}{2}$, θ may be $30°$, $150°$, $390°$, $510°$, $-330°$, etc., or, in general, $n\pi + (-1)^n 30°$.

In the solution of problems involving inverse functions,

the numerically least of these angles, called the *principal value*, is always used; *i.e.* we understand that $\sin^{-1} a$, $\tan^{-1} a$, are angles between $+90°$ and $-90°$, while the limits of $\cos^{-1} a$ are $0°$ and $180°$.

Thus, $\sin^{-1}\frac{1}{2} = 30°$, $\sin^{-1}(-\frac{1}{2}) = -30°$, $\cos^{-1}\frac{1}{2} = 60°$, $\cos^{-1}(-\frac{1}{2}) = 120°$.

ORAL WORK.

How many degrees in each of the following angles? How many radians?

1. $\cos^{-1}\frac{\sqrt{3}}{2}$?
2. $\tan^{-1} 1$?
3. $\cot^{-1}(-\sqrt{3})$?
4. $\sin^{-1}(-\frac{1}{2}\sqrt{2})$?
5. $\cos^{-1}(-\frac{1}{2}\sqrt{2})$?
6. $\sin^{-1}\left(-\frac{\sqrt{3}}{2}\right)$?
7. $\tan^{-1}\sqrt{3}$?
8. $\cos^{-1} 0$?
9. $\sin^{-1} 1$?
10. $\tan^{-1} 0$?
11. $\tan^{-1}(-1)$?
12. $\sin^{-1}(-1)$?

Find the values of the functions:

13. $\sin(\tan^{-1}\frac{1}{3}\sqrt{3})$.
14. $\tan(\cos^{-1} 1)$.
15. $\tan(\cot^{-1}[-\infty])$.
16. $\cos(\tan^{-1}\infty)$.
17. $\sin(\sin^{-1}\frac{1}{2}\sqrt{2})$.
18. $\tan(\tan^{-1} x)$.
19. $\cos(\sin^{-1} 0)$.
20. $\sin(\cos^{-1}[-1])$.
21. $\cos(\cot^{-1}\sqrt{3})$.
22. $\tan(\sin^{-1}[-1])$.
23. $\sin(\tan^{-1}[-1])$.

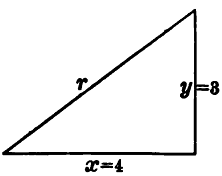


FIG. 24.

Ex. 1. Construct $\cot^{-1}\frac{4}{3}$.

Construct the right triangle xyr, so that $x = 4$, $y = 3$, whence angle $xr = \cot^{-1}\frac{4}{3}$.

2. Find $\cos(\tan^{-1}\frac{8}{15})$.

Let $\theta = \tan^{-1}\frac{8}{15}$, whence

$$\tan\theta = \tfrac{8}{15}, \text{ and } \cos\theta = \tfrac{15}{17}.$$

$$\therefore \cos\theta = \cos(\tan^{-1}\tfrac{8}{15}) = \tfrac{15}{17}.$$

3. If $\theta = \csc^{-1} a$, prove $\theta = \cos^{-1}\frac{\sqrt{a^2-1}}{a}$.

$$\csc\theta = a;\ \therefore \sin\theta = \frac{1}{a},$$

and $$\cos\theta = \sqrt{1-\frac{1}{a^2}} = \frac{\sqrt{a^2-1}}{a}, \text{ or } \theta = \cos^{-1}\frac{\sqrt{a^2-1}}{a}.$$

EXAMPLES.

1. Construct $\sin^{-1}\frac{2}{3}$, $\tan^{-1}\frac{5}{12}$, $\cos^{-1}(-\frac{1}{2})$.

2. Find $\tan(\sin^{-1}\frac{5}{13})$, $\sin(\tan^{-1}\frac{5}{12})$.

3. If $\theta = \sin^{-1} a$, prove $\theta = \tan^{-1}\dfrac{a}{\sqrt{1-a^2}}$.

4. Show that $\sin^{-1} a = 90° - \cos^{-1} a$.

5. Prove $\tan^{-1}\sqrt{3} + \cot^{-1}\sqrt{3} = \dfrac{\pi}{2}$.

6. Prove $\tan^{-1}\left(\sin\dfrac{\pi}{2}\right) = \cos^{-1}\frac{1}{2}\sqrt{2}$.

7. What angles, less than 360°, have the same tangent numerically as 10°?

8. Given $\tan 143° 22' = -0.74357$; find, correct to 0.00001, sine and cosine.

9. If $\cot^2(90° + \beta) + \csc(90° - \beta) - 1 = 0$, find $\tan\beta$.

10. Find all positive values of x, less than 360°, when $\sin x = \sin 22° 30'$; when $\tan 2x = \tan 60°$.

11. When is $\sin x = \dfrac{a^2 + b^2}{2ab}$ possible, and when impossible?

12. Verify $\sin^{-1}\frac{1}{2} + \cos^{-1}\dfrac{\sqrt{3}}{2} + \tan^{-1}\sqrt{3} = \sin^{-1}\dfrac{\sqrt{3}}{2}$.

13. What values of x will satisfy $\sin^{-1}(x^2 - x) = 30°$?

14. If $\tan^2\theta - \sec^2\alpha = 1$, prove $\sec\theta + \tan^3\theta\csc\theta = (3 + \tan^2\alpha)^{\frac{3}{2}}$.

15. Prove $\sin A(1 + \tan A) + \cos A(1 + \cot A) = \sec A + \csc A$.

16. Solve the simultaneous equations:

$$\sin^{-1}(2x + 3y) = 30° \text{ and } 3x + 2y = 2.$$

17. Verify (a) $\tan 60° = \sqrt{\dfrac{1 - \cos 120°}{1 + \cos 120°}}$.

(b) $\cos 60° = \dfrac{1 - \tan^2 30°}{1 + \tan^2 30°}$.

(c) $2\sin^2 60° = 1 - \cos 120°$.

18. Show that the cosine of the complement of $\dfrac{\pi}{6}$ equals the sine of the supplement of $\dfrac{\pi}{6}$.

REVIEW.

Before leaving a problem the student should review and master all principles involved.

1. Construct $\cos^{-1}\frac{8}{17}$; $\sin^{-1}(-\frac{3}{4})$; $\tan^{-1}2$.

2. Find $\cos(\sin^{-1}\frac{3}{5})$; $\tan(\cos^{-1}[-\frac{1}{2}])$.

3. Prove $\cot^{-1}a = \cos^{-1}\dfrac{a}{\sqrt{1+a^2}}$.

4. Given $\alpha = \cot^{-1}\frac{3}{5}$, find $\tan\alpha + \sin(90^\circ + \alpha)$.

5. Find $\tan\left(\sin^{-1}\frac{1}{2} + \cos^{-1}\dfrac{\sqrt{3}}{2}\right)$.

6. State the fundamental relations between the trigonometric functions in terms of the inverse functions. Thus,

$$\sin^{-1}a = \csc^{-1}\frac{1}{a},\quad \sin^{-1}a = \cos^{-1}\sqrt{1-a^2}, \text{ etc.}$$

7. Find all the angles, less than 360°, whose cosine equals sin 120°.

8. Given $\cot^{-1}2.8449$, find the sine and cosine of the angle, correct to 0.0001.

9. If $\tan^2(180^\circ - \theta) - \sec(180^\circ + \theta) = 5$, find $\cos\theta$.

10. If $\sin\theta = \frac{2}{3}$, find $\dfrac{\tan^2\theta + \cos^2\theta}{\tan^2\theta - \cos^2\theta}$.

11. Is $\sin x - 2\cos x + 3\sin x - 6 = 0$ a possible equation?

12. Verify (*a*) $\sin 60^\circ = \dfrac{2\tan 30^\circ}{1 + \tan^2 30^\circ}$.

(*b*) $2\cos^2 60^\circ = 1 + \cos 120^\circ$.

(*c*) $\cos 60^\circ - \cos 90^\circ = 2\cos^2 30^\circ - 2\cos^2 45^\circ$.

13. If $\sin x = \dfrac{a(a+2b)}{a^2 + 2ab + 2b^2}$, find $\sec x$ and $\tan x$.

14. Prove $\dfrac{1 + \sin\theta - \cos\theta}{1 + \sin\theta + \cos\theta} + \dfrac{1 + \sin\theta + \cos\theta}{1 + \sin\theta - \cos\theta} = 2\csc\theta$.

15. Prove

$$\cos 45^\circ + \cos 135^\circ + \cos 30^\circ + \cos 150^\circ - \cos 210^\circ + \cos 270^\circ = \sin 60^\circ.$$

16. If $\tan\theta = \dfrac{b}{\sqrt{a^2 - b^2}}$, prove that

$$\sin\theta(1 + \tan\theta) + \cos\theta(1 + \cot\theta) - \sec\theta = \frac{a}{b}.$$

17. Solve $\sin^2 x + \sin^2(x + 90^\circ) + \sin^2(x + 180^\circ) = 1$.

18. Given $\cos^2\alpha = m\sin\alpha - n$, find $\sin\alpha$.

19. If $\sin^2\beta = \dfrac{3}{2\sec\beta}$, find β.

20. Given $\tan 238° = 1.6$, find $\sin 148°$.

21. Prove $\tan^{-1} m + \cot^{-1} m = 90°$.

22. Find $\sin(\sin^{-1}p + \cos^{-1}p)$.

23. Solve $\cot^2\theta\,(2\csc\theta - 3) + 3(\csc\theta - 1) = 0$.

24. Prove $\sin^2\alpha\sec^2\beta + \tan^2\beta\cos^2\alpha = \sin^2\alpha + \tan^2\beta$.

25. Prove $\cos^6 V + \sin^6 V = 1 - 3\sin^2 V + 3\sin^4 V$.

26. What values of A satisfy $\sin 2A = \cos 3A$?

27. If $\tan C = \dfrac{\sqrt{1-m^2}}{m}$, and $\tan D = \sqrt{\dfrac{1-\cos C}{1+\cos C}}$, find $\tan D$ in terms of m.

28. If $\sin x - \cos x + 4\cos^2 x = 2$, find $\tan x$; $\sec x$.

29. Does the value of $\sec x$, derived from $\sec^2 x = \dfrac{1-2\cos^2 x}{1-\cos^2 x}$, give a possible value of x?

30. Prove

$$[\cot(90° - A) - \tan(90° + A)]\,[\sin(180° - A)\sin(90° + A)] = 1.$$

31. Prove $(1+\sin A)^2[\cot A + 2\sec A(1 - \csc A)] + \csc A\cos^3 A = 0$.

32. Given $\sin x = m\sin y$, and $\tan x = n\tan y$, find $\cos x$ and $\cos y$.

33. Given $\cot 201° = 2.6$, find $\cos 111°$.

34. Find the value of

$$\cos^{-1}\tfrac{1}{2} + \sin^{-1}\tfrac{1}{2}\sqrt{2} + \csc^{-1}(-1) + \tan^{-1}1 - 2\cot^{-1}\sqrt{3}.$$

35. Solve $2\cos^2\theta + 11\sin\theta - 7 = 0$.

36. Prove

$$\cos^2 B + \cos^2(B + 90°) + \cos^2(B + 180°) + \cos^2(B + 270°) = 2.$$

CHAPTER IV.

COMPUTATION TABLES.

33. Natural functions. It has been noted that the trigonometric functions of angles are *numbers*, but the values were found for only a few angles, viz. $0°$, $30°$, $45°$, $60°$, $90°$, etc. In computations, however, it is necessary to know the values of the functions of any angle, and tables have been prepared giving the numerical values of the functions of all angles between $0°$ and $90°$ to every minute. In these tables the functions of any given angle, and conversely the angle corresponding to any given function, can be found to any required degree of accuracy; *e.g.* by looking in the tables we find $\sin 24° 26' = 0.41363$, and also $1.6415 = \tan 58° 39'$. These numbers are called the *natural functions*, as distinguished from their logarithms, which are called the *logarithmic functions* of the angles.

Ex. 1. Find from the tables of natural functions:

$\sin 35° 14'$; $\cos 54° 46'$; $\tan 78° 29'$; $\cos 112° 58'$; $\sin 135°$.

2. Find the angles less than 180° corresponding to:

$\sin^{-1} 0.37865$; $\cos^{-1} 0.37865$; $\tan^{-1} 0.58670$; $\cos^{-1} 0.00291$; $\sin^{-1} 0.99999$.

34. Logarithms. The arithmetical processes of multiplication, division, involution, and evolution, are greatly abridged by the use of tables of logarithms of numbers and of the trigonometric ratios, which are numbers. The principles involved are illustrated in the following table:

Write in parallel columns a geometrical progression having the ratio 2, and an arithmetical progression having the difference 1, as follows:

G. P.	A. P.
1	0
2	1
4	2
8	3
16	4
32	5
64	6
128	7
256	8
512	9
1024	10
2048	11
4096	12
8192	13
16384	14
32768	15
65536	16
131072	17
262144	18
524288	19
1048576	20

It will be perceived that the numbers in the second column are the indices of the powers of 2 producing the corresponding numbers in the first column, thus: $2^6 = 64$, $2^{11} = 2048$, $2^{18} = 262144$, etc. The use of such a table will be illustrated by examples.

Ex. 1. Multiply 8192 by 128.

From the table, $8192 = 2^{13}$, $128 = 2^7$. Then by actual multiplication, $8192 \times 128 = 1048576$, or by the law of indices, $2^{13} \times 2^7 = 2^{20} = 1048576$ (from table).

Notice that the simple operation of addition is substituted for multiplication by adding the numbers in the second column opposite the given factors in the first column. This sum corresponds to the number in the first column which is the required product.

2. Divide 16384 by 512.

$16384 \div 512 = 32$, which corresponds to the result obtained by use of the table, or $2^{14} \div 2^9 = 2^5 = 32$. The operation of subtraction takes the place of division.

3. Find $\sqrt[6]{262144}$.

$$\sqrt[6]{262144} = \sqrt[6]{2^{18}} = 2^{\frac{18}{6}} = 2^3 = 8.$$

In the table, 262144 is opposite 18. $18 \div 6 = 3$, which is opposite 8, the required root; *i.e.* simple division takes the place of the tedious process of evolution.

4. Cube 64.

5. Multiply 256 by 4096.

6. Find $\sqrt[5]{32768}$.

7. Divide 1048576 by 32768.

35. The above table can be made as complete as desired by continually inserting between successive numbers in the first column the geometrical mean, and between the opposite numbers in the second, the arithmetical mean, but in practice logarithms are computed by other methods. The numbers in the second column are called the *logarithms* of the numbers opposite in the first column. 2 is called the *base* of this system, so that the *logarithm of a number is the exponent by which the base is affected to produce the number.*

Thus, the logarithm of 512 to the base 2 is 9, since $2^9 = 512$.

Logarithms were invented by a Scotchman, John Napier, early in the seventeenth century, but his method of constructing tables was different from the above. See *Encyc. Brit.*, art. "*Logarithms,*" for an exceedingly interesting account. De Morgan says that by the aid of logarithms the labor of computing has been reduced for the mathematician to about one-tenth part of the previous expense of time and labor, while Laplace has said that John Napier, by the invention of logarithms, lengthened the life of the astronomer by one-half.

Columns similar to those above might be formed with any other number as base. For practical purposes, however, 10 is always taken as the base of the system, called the *common system*, in distinction from the *natural system*, of which the base is 2.71828 ⋯, the value of the exponential series (*Higher Algebra*). The natural system is used in theoretical discussions. It follows that *common logarithms* are *indices, positive or negative, of the powers of 10.*

Thus, $10^3 = 1000$; *i.e.* $\log 1000 = 3$;

$$10^{-2} = \frac{1}{10^2} = 0.01; \text{ i.e. } \log 0.01 = -2.$$

36. Characteristic and mantissa. Clearly most numbers are not integral powers of 10. Thus 300 is more than the second and less than the third power of 10, so that

$$\log 300 = 2 \text{ plus a decimal.}$$

Evidently the logarithms of numbers generally consist of an integral and a decimal part, called respectively the *characteristic* and the *mantissa* of the logarithms.

37. Characteristic law. The characteristic of the logarithm of a number is *independent* of the digits composing the number, but *depends* on the position of the decimal point, and *is found by counting the number of places the first significant figure in the number is removed from the units' place, being positive or negative according as the first significant*

figure is at the left or the right of units' place. This follows from the fact that common logarithms are indices of powers of 10, and that 10^n, n being a positive integer, contains $n+1$ places, while 10^{-n} contains $n-1$ zeros at the right of units' place. Thus in 146.043 the first significant figure is two places at the left of units' place; the characteristic of log 146.043 is therefore 2. In 0.00379 the first significant digit is three places at the right of units' place, and the characteristic of log 0.00379 is -3.

To avoid the use of negative characteristics, such characteristics are increased by 10, and -10 is written after the logarithm. Thus, instead of $\log 0.00811 = \bar{3}.90902$, write $7.90902 - 10$. In practice the -10 is generally not written, but it must always be remembered and accounted for in the result.

Ex. Determine the characteristic of the logarithm of:

1; 46; 0.009; 14796.4; 230.001; $10^5 \times 76$; 0.525; 1.03; 0.000426.

38. Mantissa law. The mantissa of the logarithm of a number is *independent* of the position of the decimal point, but *depends* on the digits composing the number, *is always positive*, and *is found* in the tables.

For, moving the decimal point multiplies or divides a number by an integral power of 10, *i.e.* adds to or subtracts from the logarithm an integer, and hence does not affect the mantissa. Thus,

$$\begin{aligned}
\log 225.67 &= \log 225.67, \\
\log 2256.7 &= \log 225.67 \times 10^1 = \log 225.67 + 1, \\
\log 22567.0 &= \log 225.67 \times 10^2 = \log 225.67 + 2, \\
\log 22.567 &= \log 225.67 \times 10^{-1} = \log 225.67 + (-1), \\
\log 0.22567 &= \log 225.67 \times 10^{-3} = \log 225.67 + (-3),
\end{aligned}$$

so that the mantissæ of the logarithms of all numbers composed of the digits 22567 in that order are the same, .35347. Moving the decimal point affects the characteristic only. *The student must remember that the mantissa is always positive.*

Log 0.0022567 is never written $-3+.35347$, but $\bar{3}.35347$, the minus sign being written above to indicate that the characteristic alone is negative. In computations negative characteristics are avoided by adding and subtracting 10, as has been explained.

39. We may now define the *logarithm of a number as the index of the power to which a fixed number, called the base, must be raised to produce the given number.*

Thus, $a^x = b$, and $x = \log_a b$ (where $\log_a b$ is read logarithm of b to the base a) are equivalent expressions. The relation between base, logarithm, and number is always

$$(\text{base})^{\log} = \text{number}.$$

To illustrate: $\log_2 8 = 3$ is the same as $2^3 = 8$; $\log_3 81 = 4$ and $3^4 = 81$ are equivalent expressions; and so are $\log_{10} 1000 = 3$ and $10^3 = 1000$, and $\log_{10} 0.001 = -3$ and $10^{-3} = 0.001$.

Find the value of :

$$\log_4 64;\ \log_5 125;\ \log_3 243;\ \log_a (a)^{\frac{4}{5}};\ \log_{27} 3;\ \log_x 1.$$

40. From the definition it follows that the laws of indices apply to logarithms, and we have:

I. *The logarithm of a product equals the sum of the logarithms of the factors.*

II. *The logarithm of a quotient equals the logarithm of the dividend minus the logarithm of the divisor.*

III. *The logarithm of a power equals the index of the power times the logarithm of the number.*

IV. *The logarithm of a root equals the logarithm of the number divided by the index of the root.*

For if $a^x = n$ and $a^y = m$,

then $n \times m = a^{x+y}$, $\therefore \log nm = x + y = \log n + \log m$;

and $n \div m = a^{x-y}$, $\therefore \log \dfrac{n}{m} = x - y = \log n - \log m$;

also $n^r = (a^x)^r = a^{rx}$, $\therefore \log n^r = rx = r \times \log n$;

finally, $\sqrt[r]{n} = \sqrt[r]{a^x} = a^{\frac{x}{r}}$, $\therefore \log \sqrt[r]{n} = \dfrac{x}{r} = \dfrac{1}{r} \log n$.

EXAMPLES.

Given $\log 2 = 0.30103$, $\log 3 = 0.47712$, $\log 5 = 0.69897$, find:

1. $\log 4$.
2. $\log 6$.
3. $\log 10$.
4. $\log 9$.
5. $\log 25$.
6. $\log \sqrt{3}$.
7. $\log 15^3$.
8. $\log \frac{8}{3}$.
9. $\log 15 \times 9$.
10. $\log \sqrt{\frac{72}{25}}$.
11. $\log \sqrt{\dfrac{9^2 \times 5^3}{2^4 \times 10}}$.

USE OF TABLES.

41. To find the logarithm of a number.

First. Find the characteristic, as in Art. 37.

Second. Find the mantissa in the tables, thus:

(a) When the number consists of not more than four figures.

In the column N of the tables find the first three figures, and in the row N the fourth figure of the number. The mantissa of the logarithm will be found in the row opposite the first three figures and in the column of the fourth figure.

Illustration. Find $\log 42.38$.

The characteristic is 1. (Why?)

In the table in column N find the figures 423, and on the same page in row N the figure 8. The last three figures of the mantissa, 716, lie at the intersection of column 8 and row 423. To make the tables more compact the first two figures of the mantissa, 62, are printed in column 0 only. Then $\log 42.38 = 1.62716$.

$$\text{Find} \qquad \begin{aligned} &\log 0.8734 = \bar{1}.94121, \\ &\log 3.5 \quad\;\; = \log 3.500 = 0.54407, \\ &\log 36350 \;\, = 4.56050. \end{aligned}$$

(b) When the number consists of more than four figures.

Find the mantissa of the logarithm of the number composed of the first four figures as above. To correct for the remaining figures we *interpolate* by means of the *principle of proportional parts*, according to which *it is assumed that, for differences small as compared with the numbers, the differences*

between several numbers are proportional to the differences between their logarithms.

The theorem is only approximately correct, but its use leads to results accurate enough for ordinary computations.

Ex. 1. To find log 89.4562.

As above, mantissa of $\log 894500 = 0.95158$,

$$\text{mantissa of } \log 894600 = 0.95163,$$

$$\therefore \log 894600 - \log 894500 = 0.00005, \textit{ called the tabular difference.}$$

Let $\log 894562 - \log 894500 = x$ hundred-thousandths.

Now, by the principle of proportional parts,

$$\frac{\log 894562 - \log 894500}{\log 894600 - \log 894500} = \frac{894562 - 894500}{894600 - 894500},$$

or $\frac{x}{5} = \frac{62}{100}$, whence $x = .62$ of $5 = 3.1$

$$\therefore \log 89.4562 = 1.95158 + 0.00003 = 1.95161,$$

all figures after the fifth place being rejected in five-place tables. If, however, the sixth place be 5 or more, it is the practice to add 1 to the figure in the fifth place. Thus, if $x = 0.0000456$, we should call it 0.00005, and add 5 to the mantissa.

2. Find log 537.0643.

To interpolate we have $x : 9 = 643 : 1000$, *i.e.* $x = 5.787$;

$$\therefore \log 537.0643 = 2.72997 + 0.00006.$$

3. Find $\log 0.0168342 = \bar{2}.22619$.

4. Find $\log 39642.7 = 4.59816$.

42. To find the number corresponding to a given logarithm.

The characteristic of the logarithm determines the position of the decimal point (Art. 37).

(*a*) If the mantissa is in the tables, the required number is found at once.

Ex. 1. Find $\log^{-1} 1.94621$ (read, the number whose logarithm is 1.94621).

The mantissa is found in the tables at the intersection of row 883 and column 5.

$$\therefore \log^{-1} 1.94621 = 88.35,$$

the characteristic 1 showing that there are two integral places.

(*b*) If the exact mantissa of the given logarithm is not in the tables, the first four figures of the corresponding number are found, and to these are annexed figures found by interpolating by means of the principle of proportional parts, as follows:

Find the two successive mantissæ between which the given mantissa lies. Then, by the principle of proportional parts, the amount to be added to the four figures already found is such a part of 1 as the difference between the successive mantissæ is of the difference between the smaller of them and the given mantissa.

2. Find $\log^{-1} 1.43764$.

$$\begin{array}{rrr} \text{Mantissa of } \log 2740 & = & 0.43775 \\ \text{of } \underline{\log 2739} & \underline{=} & \underline{0.43759} \\ \text{Differences} \quad 1 & & 16 \end{array}$$

$$\begin{array}{rrr} \text{Mantissa of log required number} & = & 0.43764 \\ \text{of } \underline{\log 2739} & \underline{=} & \underline{0.43759} \\ \text{Differences} \quad x & & 5 \end{array}$$

By p. p. $\qquad x : 1 = 5 : 16$ and $x = \frac{5}{16} = 0.3125.$

Annexing these figures, $\log^{-1} 1.43764 = 27.393+$.

3. Find $\log^{-1} \bar{1}.48762$.

The differences in logarithms are 14, 6.

$$\therefore\ x = \frac{6}{14} = .428+,$$

$$\text{and } \log^{-1} \bar{1}.48762 = 0.30744+.$$

4. Find $\log 891.59$; $\log 0.023$; $\log \frac{1}{2}$; $\log 0.1867$; $\log \sqrt{2}$.

5. Find $\log^{-1} 2.21042$; $\log^{-1} 0.55115$; $\log^{-1} \bar{1}.89003$.

43. Logarithms of trigonometric functions. These might be found by first taking from the tables the natural functions of the given angle, and then the logarithms of these numbers. It is more expeditious, however, to use tables showing directly the logarithms of the functions of angles less than 90° to every minute. Functions of angles greater than 90° are reduced to functions of angles less than 90° by

the formulæ of Art. 29. To make the work correct for seconds, or any fractional part of a minute, interpolation is necessary by the principle of proportional parts, thus :

Ex. 1. Find $\log \sin 28° 32' 21''$.

In the table of logarithms of trigonometric functions, find 28° at the top of the page, and in the minute column at the left find 32′. Then under log sin column find $\log \sin 28° 32' = 9.67913 - 10$

$$\log \sin \underline{28° 33 \;= 9.67936 - 10}$$

$$\text{Differences} \quad 1' \qquad 23$$

By p. p. $\qquad x : 23 = 21'' : 60''$, *i.e.* $x = \frac{21}{60} \times 23 = 8.05.$

$$\therefore \log \sin 28° 32' 21'' = 9.67913 + 0.00008 - 10$$
$$= 9.67921 - 10.$$

Whenever functions of angles are less than unity, *i.e.* are decimals (as sine and cosine always are, except when equal to unity, and as tangent is for angles less than 45°), the characteristic of the logarithm will be negative, and, accordingly, 10 is always added in the tables, and it must be remembered that 10 is to be subtracted. Thus, in the example above, the characteristic of the logarithm is not 9, but $\bar{1}$, and the logarithm is not 9.67913, as written in the tables, but 9.67913 − 10.

2. Find $\log \cos 67° 27' 50''$.

In the table of logarithms at the foot of the page, find 67°, and in the minute column at the right, 27′. Then computing the difference as above, $x = 25$.

But it must be noted that cosine decreases as the angle increases toward 90°. Hence, $\log \cos 67° 27' 50''$ is less than $\log \cos 67° 27'$, *i.e.* the difference 25 must be subtracted, so that

$$\log \cos 67° 27' 50'' = 9.58375 - 0.00025 - 10$$
$$= 9.58350 - 10.$$

44. To find the angle when the logarithm is given, find the successive logarithms between which the given logarithm lies, compute by the principle of proportional parts the seconds, and add them to the less of the two angles corresponding to the successive logarithms. This will not necessarily be the angle corresponding to the less of the two logarithms; for, as has been seen, the number, and, therefore, the logarithm, may decrease as the angle increases.

Ex. 1. Find the angle whose log tan is 9.88091.

$$\begin{array}{rl} \log\tan 37^\circ 14' &= 9.88079 - 10 \\ \log\tan 37^\circ 15' &= 9.88105 - 10 \\ \hline \text{Differences} \quad 60'' & \quad 26 \end{array}$$

$$\begin{array}{rl} \log\tan 37^\circ 14' &= 9.88079 - 10 \\ \log\tan \text{angle required} &= 9.88091 - 10 \\ \hline \text{Differences} \quad x'' & \quad 12 \end{array}$$

$\therefore\ x : 60 = 12 : 26$, or $x'' = \frac{12}{26} \times 60'' = 28''$, approximately, and the angle is $37^\circ 14' 28''$.

2. Find the angle whose $\log\cos = 9.82348$.

We find $x = \frac{6}{14} \times 60'' = 26''$, and the angle is $48^\circ 14' 26''$.

3. Show that
$$\log\cos\ 25^\circ 31' 20'' = 9.95541;$$
$$\log\sin 110^\circ 25' 20'' = 9.97181;$$
$$\log\tan\ 49^\circ 52' 10'' = 0.07417.$$

4. Show that the angle whose log tan is 9.92501 is $40^\circ 4' 39''$; whose log sin is 9.88365 is $49^\circ 54' 18''$; whose log cos is 9.50828 is $71^\circ 11' 49''$.

45. Cologarithms. In examples involving multiplications and divisions it is more convenient, if n is any divisor, to add $\log\frac{1}{n}$ than to subtract $\log n$. The logarithm of $\frac{1}{n}$ is called the cologarithm of n. Since

$$\log\frac{1}{n} = \log 1 - \log n = 0 - \log n,$$

it follows that $\text{colog}\, n = -\log n$, *i.e.* $\log n$ subtracted from zero. To avoid negative results, add and subtract 10.

Ex. 1. Find colog 2963.

$$\begin{array}{rl} \log 1 &= 10.00000 - 10 \\ \log 2963 &= \ \ 3.47173 \\ \hline \therefore\ \text{colog}\, 2963 &= \ \ 6.52827 - 10 \end{array}$$

2. Find colog tan $16^\circ 17'$.

$$\begin{array}{rl} \log 1 &= 10.00000 - 10 \\ \log\tan 16^\circ 17' &= \ \ 9.46554 - 10 \\ \hline \therefore\ \text{colog}\tan 16^\circ 17' &= \ \ 0.53446 \end{array}$$

By means of the definitions of the trigonometric functions, the parts of a right triangle may be computed if any two parts, one of them being a side, are given. Thus,

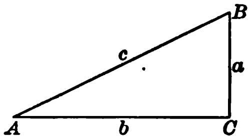


FIG. 25.

given a and A in the rt. triangle ABC.

Then $c = a \div \sin A,\ b = a \div \tan A,$

and $B = 90° - A.$

Again, if a and b are given, then

$\tan A = \frac{a}{b},\ c = a \div \sin A,$ and $B = 90° - A.$

3. Given $c = 25.643$, $B = 37°\ 25'\ 20''$, compute the other parts.

$$A = 90° - 37°\ 25'\ 20'' = 52°\ 34'\ 40''.$$

$a = c \cos B.$	$b = a \tan B.$
$\log c = 1.40897$	$\log a = 1.30889$
$\log \cos B = 9.89992$	$\log \tan B = 9.88376$
$\log a = 1.30889$	$\log b = 1.19265$
$\therefore\ a = 20.365.$	$\therefore\ b = 15.583.$

Check: $c^2 = a^2 + b^2 = 20.365^2 + 15.583^2 = 657.57 = 25.643^2.$

4. Given $b = 0.356$, $B = 63°\ 28'\ 40''$, compute the other parts.

$$A = 26°\ 31'\ 20''.$$

$c = \dfrac{b}{\sin B}.$	$a = \dfrac{b}{\tan B}.$
$\log b = 9.55145$	$\log b = 9.55145$
$\text{colog} \sin B = 0.04829$	$\text{colog} \tan B = 9.69816$
$\log c = 9.59974$	$\log a = 9.24961$
$c = 0.3979$	$a = 0.1777$

Check: $c^2 - a^2 = 0.1583 - 0.03157 = 0.12673 = b^2.$

EXAMPLES.

Compute the other parts:

1. Given $a = 9.325$, $A = 43°\ 22'\ 35''$.
2. Given $c = 240.32$, $a = 174.6$.
3. Given $B = 76°\ 14'\ 23''$, $a = 147.53$.
4. Given $a = 2789.42$, $b = 4632.19$.
5. Given $c = 0.0213$, $A = 23°\ 14''$.
6. Given $b = 2$, $c = 3$.

CHAPTER V.

APPLICATIONS.

46. Many problems in measurements of heights and distances may be solved by applying the preceding principles. By means of instruments certain distances and angles may be measured, and from the data thus determined other distances and angles computed. The most common instruments are the *chain*, the *transit*, and the *compass*.

The *chain* is used to measure distances. Two kinds are in use, the *engineer's chain* and the *Gunter's chain*. They each contain 100 links, each link in the engineer's chain being 12 inches long, and in the Gunter's 7.92 inches.

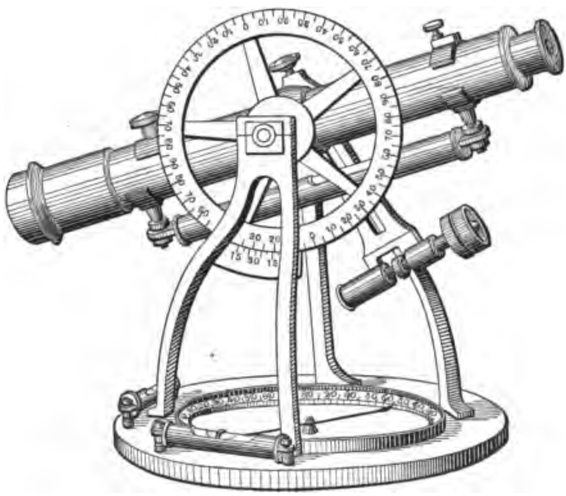


FIG. 26.

The *transit* is the instrument most used to measure horizontal angles, and with certain attachments to measure vertical angles. The figure shows the form of the instrument.

The *mariner's compass* is used to determine the directions, or *bearings*, of objects at sea. Each quadrant is divided into 8 parts, making the 32 points of the compass, so that each point contains $11° 15'$.

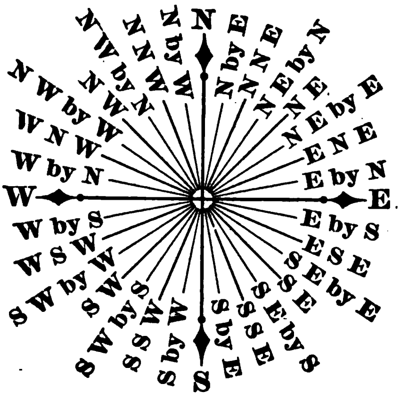


FIG. 27.

47. The angle between the horizontal plane and the line of vision from the eye to the object is called the *angle of elevation*, or of *depression*, according as the object is above or below the observer.

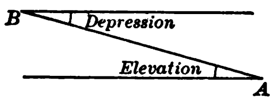


FIG. 28.

It is evident that the elevation angle of B, as seen from A, is equal to the depression angle of A, as seen from B, so that in the solution of examples the two angles are interchangeable.

PROBLEMS.

48. Some of the more common problems met with in practice are illustrated by the following:

To find the height of an object when the foot is accessible.

The distance BC, and the elevation angle B are measured, and x is determined from the relation $x = BC \tan B$.

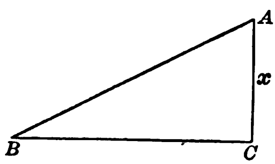


FIG. 29.

Ex. 1. The elevation angle of a cliff measured from a point 300 ft. from its base is found to be 30°. How high is the cliff?

$$BC = 300,\ B = 30°.$$

Then $$x = 300 \cdot \tan 30° = 300 \cdot \tfrac{1}{3}\sqrt{3} = 100\sqrt{3}.$$

2. From a point 175 ft. from the foot of a tree the elevation of the top is found to be 27° 19′. Find the height of the tree.

The problem may be solved by the use of natural functions, or of logarithms. The work should be arranged for the solution before the tables are opened. Let the student complete.

$$BC = 175.\quad B = 27°\ 19'.$$

Then $x = BC \tan B.$ Or by natural functions,

$$\log BC =$$
$$\log \tan B =$$
$$\log x =$$
$$\therefore x = 90.39.$$

$$BC = 175$$
$$\tan B = 0.5165$$
$$\therefore x = 90.3875.$$

To find the height of an object when the foot is inaccessible.

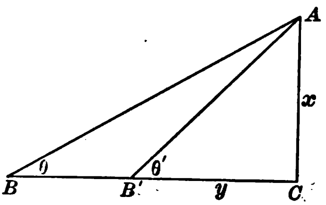


FIG. 30.

Measure BB', θ and θ'.

Then $$x = \frac{BC}{\cot\theta} = \frac{BB' + B'C}{\cot\theta}.$$

But $B'C = x \cot\theta'$, whence substituting,

$$x = \frac{BB'}{\cot\theta - \cot\theta'},$$

which is best solved by the use of the natural functions of θ and θ'.

3. Measured from a certain point at its base the elevation of the peak of a mountain is 60°. At a distance of one mile directly from this point the elevation is 30°. Find the height of the mountain.

$$BB' = 5280 \text{ ft.},\quad \theta = 30°,\quad \theta' = 60°.$$

$$x = \frac{y + 5280}{\cot 30°}.\quad \text{But } y = x \cot 60°.$$

$$\therefore x = \frac{5280}{\cot 30° - \cot 60°} = 4572.48 \text{ ft.}$$

In surveying it is often necessary to make measurements across a stream or other obstacle too wide to be spanned by a single chain.

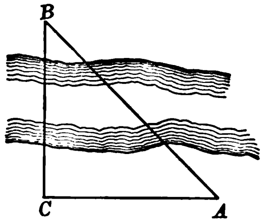


FIG. 31.

To find the distance from C to a point B on the opposite side of a stream.

At C measure a right angle, and take CA a convenient distance. Measure angle A, then

$$BC = CA \cdot \tan A.$$

4. Find CB when angle $A = 47° 16'$, and $CA = 250$ ft.

5. From a point due south of a kite its elevation is found to be 42° 30′; from a point 20 yds. due west from this point the elevation is 36° 24′. How high is the kite above the ground?

$$AB = x \cdot \cot 42° 30',$$

$$AC = x \cdot \cot 36° 24',$$

$$AC^2 - AB^2 = BC^2 = 400.$$

$$\therefore x^2 (\cot^2 36° 24' - \cot^2 42° 30') = 400,$$

whence

$$x^2 = \frac{400}{.6489}, \text{ and } x = \frac{20}{.805} = 24.84 \text{ yds.}$$

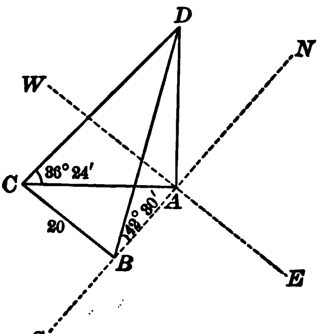


FIG. 32.

EXAMPLES.

1. What is the altitude of the sun when a tree 71.5 ft. high casts a shadow 37.75 ft. long?

2. What is the height of a balloon directly over Ann Arbor when its elevation at Ypsilanti, 8 miles away, is 10° 15′?

3. The Washington monument is 555 ft. high. How far apart are two observers who, from points due east, see the top of the monument at elevations of 23° 20′ and 47° 30′, respectively?

4. A mountain peak is observed from the base and top of a tower 200 ft. high. The elevation angles being 25° 30′ and 23° 15′, respectively, compute the height of the mountain above the base of the tower.

5. From a point in the street between two buildings the elevation angles of the tops of the buildings are 30° and 60°. On moving across

the street 20 ft. toward the first building the elevation angles are found to be each 45°. Find the width of the street and the height of each building.

6. From the peak of a mountain two towns are observed due south. The first is seen at a depression of 48° 40′, and the second, 8 miles farther away and in the same horizontal plane, at a depression of 20° 50′. What is the height of the mountain above the plane?

7. A building 145 ft. long is observed from a point directly in front of one corner. The length of the building subtends $\tan^{-1} 3$, and the height $\tan^{-1} 2$. Find the height.

8. An inaccessible object is observed to lie due N.E. After the observer has moved S.E. 2 miles, the object lies N.N.E. Find the distance of the object from each point of observation.

9. Assuming the earth to be a sphere with a radius of 3963 miles, find the height of a lighthouse just visible from a point 15 miles distant at sea.

10. The angle of elevation of a tower 120 ft. high due north of an observer was 35°; what will be its angle of elevation from a point due west from the first point of observation 250 ft.? Also the distance of the observer from the base of the tower in each position?

11. A railway 5 miles long has a uniform grade of 2° 30′; find the rise per mile. What is the grade when the road rises 70 ft. in one mile?

(The grade depends on the tangent of the angle.)

12. The foot of a ladder is in the street at a point 30 ft. from the line of a building, and just reaches a window $22\frac{1}{2}$ ft. above the ground. By turning the ladder over it just reaches a window 36 ft. above the ground on the other side of the street. Find the breadth of the street.

13. From a point 200 ft. from the base of the Forefathers' monument at Plymouth, the base and summit of the statue of Faith are at an elevation of 12° 40′ 48″ and 22° 2′ 53″, respectively; find the height of the statue and of the pedestal on which it stands.

14. At a distance of 100 ft. measured in a horizontal plane from the foot of a tower, a flagstaff standing on the top of the tower subtends an angle of 8°, while the tower subtends an angle of 42° 20′. Find the length of the flagstaff.

15. The length of a string attached to a kite is 300 ft. The kite's elevation is 56° 6′. Find the height of the kite.

16. From two rocks at sea level, 50 ft. apart, the top of a cliff is observed in the same vertical plane with the rocks. The angles of elevation of the cliff from the two rocks are 24° 40′ and 32° 30′. What is the height of the cliff above the sea?

CHAPTER VI.

GENERAL FORMULÆ — TRIGONOMETRIC EQUATIONS AND IDENTITIES.

49. Thus far functions of single angles only have been considered. Relations will now be developed to express functions of angles which are sums, differences, multiples, or sub-multiples of single angles in terms of the functions of the single angles from which they are formed.

First it will be shown that,

$$\sin(\alpha \pm \beta) = \sin\alpha \cos\beta \pm \cos\alpha \sin\beta,$$

$$\cos(\alpha \pm \beta) = \cos\alpha \cos\beta \mp \sin\alpha \sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha \tan\beta}.$$

The following cases must be considered:

1. α, β, $\alpha + \beta$ acute angles.
2. α, β, acute, but $\alpha + \beta$ an obtuse angle.
3. Either α, or β, or both, of any magnitude, positive or negative.

The figures apply to cases 1 and 2.

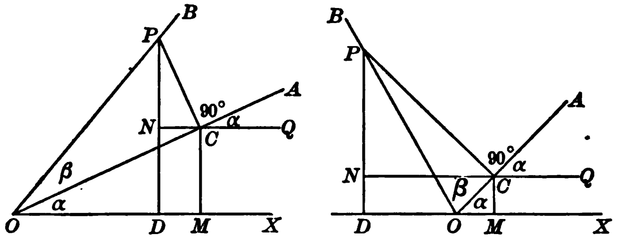


FIG. 33.

Let the terminal line revolve through the angle α, and then through the angle β, to the position OB, so that angle

$XOB = \alpha + \beta$. Through any point P in OB draw perpendiculars to the sides of α, DP and CP, and through C draw a perpendicular and a parallel to OX, MC and NC.

Then the angle $QCA = \alpha$ (why?), and CNP is the triangle of reference for angle $QCP = 90° + \alpha$.

CNP is sometimes treated as the triangle of reference for angle CPN. The fallacy of this appears when we develop $\cos(\alpha + \beta)$, in which PC would be treated as both plus and minus.

Now $\sin(\alpha + \beta) = \sin XOB = \dfrac{DP}{OP} = \dfrac{MC}{OP} + \dfrac{NP}{OP}$,

or expressing in trigonometric ratios,

$$= \frac{MC}{OC} \cdot \frac{OC}{OP} + \frac{NP}{CP} \cdot \frac{CP}{OP}$$
$$= \sin\alpha\cos\beta + \sin(90° + \alpha)\sin\beta.$$

Hence, since $\sin(90° + \alpha) = \cos\alpha$, we have

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta.$$

In like manner

$$\cos(\alpha + \beta) = \cos XOB = \frac{OD}{OP} = \frac{OM}{OP} + \frac{CN}{OP},$$

or expressing in trigonometric ratios,

$$= \frac{OM}{OC} \cdot \frac{OC}{OP} + \frac{CN}{CP} \cdot \frac{CP}{OP}$$
$$= \cos\alpha\cos\beta + \cos(90° + \alpha)\sin\beta.$$

And since $\cos(90° + \alpha) = -\sin\alpha$, we have

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta.$$

It will be noted that the wording of the demonstration applies to both figures, the only difference being that when $\alpha + \beta$ is obtuse OD is negative. CN is negative in each figure.

50. In the case, when α, or β, or both, are of any magnitude, positive or negative, figures may be constructed as before described by *drawing through any point in the terminal line of β a perpendicular to each side of α, and through the foot of the perpendicular on the terminal line of α a perpendicular and a parallel to the initial line of α.* Noting negative lines,

the demonstrations already given will be found to apply for all values of α and β.

To make the proof complete by this method would require an unlimited number of figures, *e.g.* we might take α obtuse, both α and β obtuse, either or both greater than 180°, or than 360°, or negative angles, etc.

Instead of this, however, the generality of the proposition is more readily shown algebraically, as follows:

Let $\alpha' = 90° + \alpha$ be any obtuse angle, and α, β, acute angles.

Then

$$\begin{aligned}\sin(\alpha' + \beta) &= \sin(90° + \alpha + \beta) = \cos(\alpha + \beta)\\ &= \cos\alpha\cos\beta - \sin\alpha\sin\beta\\ &= \sin(90° + \alpha)\cos\beta + \cos(90° + \alpha)\sin\beta \text{(why?)}\\ &= \sin\alpha'\cos\beta + \cos\alpha'\sin\beta.\end{aligned}$$

In like manner, considering any obtuse angle $\beta' = 90° + \beta$, it can be shown that

$$\sin(\alpha' + \beta') = \sin\alpha'\cos\beta' + \cos\alpha'\sin\beta'.$$

Show that $\cos(\alpha' + \beta') = \cos\alpha'\cos\beta' - \sin\alpha'\sin\beta'$.

By further substitutions, *e.g.* $\alpha'' = 90° \pm \alpha'$, $\beta'' = 90° \pm \beta'$, etc., it is clear that the above relations hold for all values, positive or negative, of the angles α and β.

Since α and β may have any values, we may put $-\beta$ for β, and $\sin(\alpha + [-\beta])$

$$\begin{aligned}= \sin(\alpha - \beta) &= \sin\alpha\cos(-\beta) + \cos\alpha\sin(-\beta)\\ &= \sin\alpha\cos\beta - \cos\alpha\sin\beta \text{ (why?).}\end{aligned}$$

Also $\cos(\alpha - \beta) = \cos\alpha\cos(-\beta) - \sin\alpha\sin(-\beta)$

$$= \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

Finally,

$$\tan(\alpha \pm \beta) = \frac{\sin(\alpha \pm \beta)}{\cos(\alpha \pm \beta)} = \frac{\sin\alpha\cos\beta \pm \cos\alpha\sin\beta}{\cos\alpha\cos\beta \mp \sin\alpha\sin\beta}$$

$$= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} \pm \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} \mp \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}} = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}.$$

ORAL WORK.

By the above formulæ develop:

1. $\sin(2A + 3B)$.
2. $\cos(90^\circ - B)$.
3. $\tan(45^\circ + \phi)$.
4. $\sin 2A = \sin(A + A)$.
5. $\cos 2\theta$.
6. $\tan(180^\circ + C)$.
7. $\sin 90^\circ = \sin(45^\circ + 45^\circ)$.
8. $\cos 90^\circ$.
9. $\tan 90^\circ$.
10. $\sin(90^\circ + \beta + \gamma)$.
11. $\cos(270^\circ - m - n)$.
12. $\tan(90^\circ + m + n)$.

Ex. 1. Find $\sin 75^\circ$.

$$\sin 75^\circ = \sin(45^\circ + 30^\circ) = \sin 45^\circ \cos 30^\circ + \cos 45^\circ \sin 30^\circ$$
$$= \frac{1}{\sqrt{2}} \cdot \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} \cdot \frac{1}{2} = \frac{1 + \sqrt{3}}{2\sqrt{2}} = 0.9659.$$

2. Find $\tan 15^\circ$.

$$\tan 15^\circ = \tan(45^\circ - 30^\circ) = \frac{\tan 45^\circ - \tan 30^\circ}{1 + \tan 45^\circ \tan 30^\circ}$$
$$= \frac{1 - \dfrac{1}{\sqrt{3}}}{1 + \dfrac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1} = 2 - \sqrt{3} = 0.2679.$$

3. Prove $\dfrac{\sin 3A}{\sin A} - \dfrac{\cos 3A}{\cos A} = 2.$

Combining, $\dfrac{\sin 3A \cos A - \cos 3A \sin A}{\sin A \cos A} = \dfrac{\sin(3A - A)}{\sin A \cos A}$

$$= \frac{\sin 2A}{\sin A \cos A} = \frac{\sin(A + A)}{\sin A \cos A} = \frac{\sin A \cos A + \cos A \sin A}{\sin A \cos A} = 2.$$

4. Prove $\tan^{-1} a + \tan^{-1} b = \tan^{-1} \dfrac{a + b}{1 - ab}.$

Let $\alpha = \tan^{-1} a,\ \beta = \tan^{-1} b,\ \gamma = \tan^{-1} \dfrac{a + b}{1 - ab}.$

Hence, $\tan\alpha = a,\ \tan\beta = b,\ \tan\gamma = \dfrac{a + b}{1 - ab}.$

Then $\alpha + \beta = \gamma$, and hence $\tan(\alpha + \beta) = \tan\gamma$.

Expanding, $\dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta} = \tan\gamma.$

Substituting, $\dfrac{a + b}{1 - ab} = \dfrac{a + b}{1 - ab}.$

EXAMPLES.

1. Find $\cos 15°$, $\tan 75°$.

2. Prove $\cot(\alpha \pm \beta) = \dfrac{\cot\alpha\cot\beta \mp 1}{\cot\beta \pm \cot\alpha}$.

3. Prove geometrically $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$,
and $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta$,

given (*a*) α acute, β obtuse;
(*b*) α, β, obtuse;
(*c*) α, β, either, or both, negative angles.

4. Prove geometrically $\tan(\alpha+\beta) = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$.

Verify the formula by assigning values to α and β, and finding the values of the functions from the tables of natural tangents.

5. Prove $\cos(\alpha+\beta)\cos(\alpha-\beta) = \cos^2\alpha - \sin^2\beta$.

6. Show that $\tan\alpha + \tan\beta = \dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.

7. Given $\tan\alpha = \frac{1}{2}$, $\tan\beta = \frac{1}{3}$, find $\sin(\alpha+\beta)$

8. Given $\sin 280° = s$, find $\sin 170°$.

9. If $\alpha = 67° 22'$, $\beta = 128° 40'$, by use of the tables of natural functions verify the formulæ on page 56.

10. Prove $\tan^{-1}\dfrac{\sqrt{x}+\sqrt{a}}{1-\sqrt{ax}} = \tan^{-1}\sqrt{x} + \tan^{-1}\sqrt{a}$.

11. Prove $\tan^{-1}\dfrac{2x-b}{b\sqrt{3}} + \tan^{-1}\dfrac{2b-x}{x\sqrt{3}} = \tan^{-1}\sqrt{3}$.

12. Prove $\sec^{-1}\dfrac{a}{\sqrt{a^2-x^2}} = \sin^{-1}\dfrac{x}{a}$.

13. If $\alpha+\beta=\omega$, prove $\cos^2\alpha + \cos^2\beta - 2\cos\alpha\cos\beta\cos\omega = \sin^2\omega$.

14. Solve $\frac{1}{2}\sin\theta = 1 - \cos\theta$.

15. Prove $\sin(A+B)\cos A - \cos(A+B)\sin A = \sin B$.

16. Prove $\cos(A+B)\cos(A-B) + \sin(A+B)\sin(A-B) = \cos 2B$.

17. Prove $\sin(2\alpha-\beta)\cos(\alpha-2\beta)$
$- \cos(2\alpha-\beta)\sin(\alpha-2\beta) = \sin(\alpha+\beta)$.

18. Prove $\sin(n-1)\alpha\cos(n+1)\alpha + \cos(n-1)\alpha\sin(n+1)\alpha = \sin 2n\alpha$.

19. Prove $\sin(135° - \theta) + \cos(135° + \theta) = 0$.

20. Prove $1 - \tan^2\alpha \tan^2\beta = \dfrac{\cos^2\beta - \sin^2\alpha}{\cos^2\alpha \cos^2\beta}.$

21. Prove $\dfrac{\tan\alpha + \tan\beta}{\cot\alpha + \cot\beta} = \tan\alpha \tan\beta.$

22. $\tan^2\left(\dfrac{\pi}{4} - \alpha\right) = \dfrac{1 - 2\sin\alpha\cos\alpha}{1 + 2\sin\alpha\cos\alpha}.$

51. The following formulæ are very important and should be carefully memorized. They enable us to change sums and differences to products, *i.e.* to displace terms by factors.

$$\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\sin\theta - \sin\phi = 2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2},$$

$$\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2},$$

$$\cos\theta - \cos\phi = -2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}.$$

Since $\quad \sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta,$

and $\quad \sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta,$

then $\quad \sin(\alpha+\beta) + \sin(\alpha-\beta) = 2\sin\alpha\cos\beta, \qquad (1)$

and $\quad \sin(\alpha+\beta) - \sin(\alpha-\beta) = 2\cos\alpha\sin\beta. \qquad (2)$

Also since $\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$

and $\quad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta,$

then $\quad \cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta, \qquad (3)$

and $\quad \cos(\alpha+\beta) - \cos(\alpha-\beta) = -2\sin\alpha\sin\beta. \qquad (4)$

Put $\quad \alpha+\beta = \theta$

and $\quad \alpha-\beta = \phi$

$$2\alpha = \theta + \phi, \text{ and } \alpha = \frac{\theta+\phi}{2},$$

$$2\beta = \theta - \phi, \text{ and } \beta = \frac{\theta-\phi}{2}.$$

Substituting in (1), (2), (3), (4), we have the above formulæ.

EXAMPLES.

1. Prove $\dfrac{\sin 2\theta + \sin\theta}{\cos 2\theta + \cos\theta} = \tan\dfrac{3\theta}{2}$.

By formulæ of last article the first member becomes

$$\frac{2\sin\dfrac{3\theta}{2}\cos\dfrac{\theta}{2}}{2\cos\dfrac{3\theta}{2}\cos\dfrac{\theta}{2}} = \tan\frac{3\theta}{2}.$$

2. Prove $\dfrac{\sin\alpha + 2\sin 3\alpha + \sin 5\alpha}{\sin 3\alpha + 2\sin 5\alpha + \sin 7\alpha} = \dfrac{\sin 3\alpha}{\sin 5\alpha}$.

$$\frac{(\sin\alpha + \sin 5\alpha) + 2\sin 3\alpha}{(\sin 3\alpha + \sin 7\alpha) + 2\sin 5\alpha} = \frac{2\sin 3\alpha\cos 2\alpha + 2\sin 3\alpha}{2\sin 5\alpha\cos 2\alpha + 2\sin 5\alpha}$$

$$= \frac{(\cos 2\alpha + 1)\sin 3\alpha}{(\cos 2\alpha + 1)\sin 5\alpha} = \frac{\sin 3\alpha}{\sin 5\alpha}.$$

3. Prove $\dfrac{\sin(4A - 2B) + \sin(4B - 2A)}{\cos(4A - 2B) + \cos(4B - 2A)} = \tan(A + B)$.

$$\frac{2\sin\dfrac{4A - 2B + 4B - 2A}{2}\cos\dfrac{4A - 2B - 4B + 2A}{2}}{2\cos\dfrac{4A - 2B + 4B - 2A}{2}\cos\dfrac{4A - 2B - 4B + 2A}{2}}$$

$$= \frac{\sin(A + B)}{\cos(A + B)} = \tan(A + B).$$

4. Prove $\sin 50° - \sin 70° + \sin 10° = 0$.

$$2\cos\frac{50° + 70°}{2}\sin\frac{50° - 70°}{2} = 2\cos 60°\sin(-10°) = -\sin 10°.$$

5. Prove $\dfrac{\cos 2\alpha\cos 3\alpha - \cos 2\alpha\cos 7\alpha + \cos\alpha\cos 10\alpha}{\sin 4\alpha\sin 3\alpha - \sin 2\alpha\sin 5\alpha + \sin 4\alpha\sin 7\alpha} = \cot 6\alpha\cot 5\alpha$.

By (3) and (4), p. 61,

$$\frac{\cos 5\alpha + \cos\alpha - \cos 9\alpha - \cos 5\alpha + \cos 11\alpha + \cos 9\alpha}{\cos\alpha - \cos 7\alpha - \cos 3\alpha + \cos 7\alpha + \cos 3\alpha - \cos 11\alpha}$$

$$= \frac{\cos\alpha + \cos 11\alpha}{\cos\alpha - \cos 11\alpha} = \frac{2\cos 6\alpha\cos 5\alpha}{2\sin 6\alpha\sin 5\alpha} = \cot 6\alpha\cot 5\alpha.$$

ORAL WORK.

By the formulæ of Art. 51 transform:

6. $\cos 5\alpha + \cos\alpha$.

7. $\cos\alpha - \cos 5\alpha$.

8. $2\sin 3\theta\cos\theta$.

9. $\sin 2\alpha - \sin 4\alpha$.

10. $\cos 9\theta \cos 2\theta$.

11. $\sin\theta + \sin\frac{\theta}{2}$.

12. $\sin 75° \sin 15°$.

13. $\cos 7p - \cos 2p$.

14. $\cos(2p+3q)\sin(2p-3q)$.

15. $\sin\frac{3t}{2} - \sin\frac{t}{2}$.

16. $\cos(30° + 2\phi)\sin(30° - \phi)$.

17. $\sin(2r+s) + \sin(2r-s)$.

18. $\cos(2\beta - \alpha) - \cos 3\alpha$.

19. $\sin 36° + \sin 54°$.

20. $\cos 60° + \cos 20°$.

21. $\sin 30° + \cos 30°$.

Prove:

22. $\dfrac{\sin\alpha + \sin\beta}{\sin\alpha - \sin\beta} = \tan\dfrac{\alpha+\beta}{2}\cot\dfrac{\alpha-\beta}{2}$.

23. $\dfrac{\cos\alpha + \cos\beta}{\cos\beta - \cos\alpha} = \cot\dfrac{\alpha+\beta}{2}\cot\dfrac{\alpha-\beta}{2}$.

24. $\dfrac{\sin x + \sin y}{\cos x + \cos y} = \tan\dfrac{x+y}{2}$.

25. $\dfrac{\sin x - \sin y}{\cos x - \cos y} = -\cot\dfrac{x+y}{2}$.

26. $\cos 55° + \sin 25° = \sin 85°$.

Simplify:

27. $\dfrac{\sin B + \sin 2B + \sin 3B}{\cos B + \cos 2B + \cos 3B}$.

28. $\dfrac{\sin C - \sin 4C + \sin 7C - \sin 10C}{\cos C - \cos 4C + \cos 7C - \cos 10C}$.

52. *Functions of an angle in terms of those of the half angle.*

If in $\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$, $\alpha = \beta$,

then $\sin(\alpha+\alpha) = \mathbf{\sin 2\alpha = 2\sin\alpha\cos\alpha}$.

In like manner

$$\cos(\alpha+\alpha) = \mathbf{\cos 2\alpha = \cos^2\alpha - \sin^2\alpha}$$
$$\mathbf{= 2\cos^2\alpha - 1}$$
$$\mathbf{= 1 - 2\sin^2\alpha};$$

and

$$\mathbf{\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}}.$$

ORAL WORK.

Ex. Express in terms of functions of half the given angles:

1. $\sin 4\alpha$.
2. $\cos 3p$.
3. $\tan 5t$.
4. $\cos x$.
5. $\sin\frac{\beta}{2}$.
6. $\sin(2p-q)$.
7. $\cos(30^\circ+2\phi)$.
8. $\sin(x+y)$.

9. From the functions of 30° find those of 60°; from the functions of 45°, those of 90°.

53. *Functions of an angle in terms of those of twice the angle.*

By Art. 52, $\cos\alpha = 1-2\sin^2\frac{\alpha}{2} = 2\cos^2\frac{\alpha}{2}-1.$

$$\therefore\ 2\sin^2\frac{\alpha}{2} = 1-\cos\alpha, \qquad \text{and} \quad 2\cos^2\frac{\alpha}{2} = 1+\cos\alpha.$$

$$\sin\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}; \qquad \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}}.$$

$$\therefore\ \tan\frac{\alpha}{2} = \frac{\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}.$$

Explain the significance of the $\pm$ sign before the radicals. Express in terms of the double angle the functions of 120°; 50°; 90°, with proper signs prefixed.

Ex. 1. Express in terms of functions of twice the given angles each of the functions in Examples 1–8 above.

2. From the functions of 45° find those of 22° 30′; from the functions of 36°, those of 18° (see tables of natural functions).

3. Find the corresponding functions of twice and of half each of the following angles, and verify results by the tables of natural functions:

Given $\sin 26^\circ 42' = 0.4493,$

$\tan 62^\circ 24' = 1.9128,$

$\cos 21^\circ 34' = 0.9300.$

4. Prove $\tan^{-1}\sqrt{\frac{1-\cos x}{1+\cos x}} = \frac{x}{2}.$

5. $2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}.$

6. If A, B, C are angles of a triangle, prove

$$\sin A + \sin C + \sin B = 4 \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}.$$

7. If $\cos^2 \alpha + \cos^2 2\alpha + \cos^2 3\alpha = 1$, then

$$\cos \alpha \cos 2\alpha \cos 3\alpha = 0.$$

8. Prove $\cot A - \cot 2A = \csc 2A$.

9. Prove $\dfrac{\tan\left(\frac{\pi}{4} - \frac{\phi}{2}\right)}{\tan\left(\frac{\pi}{4} + \frac{\phi}{2}\right)} = \left[\dfrac{1 - \tan\frac{\phi}{2}}{1 + \tan\frac{\phi}{2}}\right]^2.$

10. $\dfrac{\tan\alpha}{\tan(\alpha + \phi)} = 1 - \dfrac{2\sin\phi}{\sin(2\alpha + \phi) + \sin\phi}.$

11. If $y = \tan^{-1}\dfrac{\sqrt{1 + x^2} + \sqrt{1 - x^2}}{\sqrt{1 + x^2} - \sqrt{1 - x^2}}$, prove $x^2 = \sin 2y$.

12. Prove $\tan^{-1}\dfrac{\sqrt{1 + x^2} - 1}{x} + \tan^{-1}\dfrac{2x}{1 - x^2} = \dfrac{5}{2}\tan^{-1} x.$

13. If $y = \sin^{-1}\dfrac{x}{\sqrt{1 + x^2}}$, prove $x = \tan y$.

14. Prove $\cos^2 \alpha + \cos^2 \beta - 1 = \cos(\alpha + \beta)\cos(\alpha - \beta)$.

15. Prove $\sqrt{(\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2} = 2\sin\dfrac{\alpha - \beta}{2}.$

16. Prove $\sin^{-1}\sqrt{\dfrac{x}{a + x}} = \tan^{-1}\sqrt{\dfrac{x}{a}} = \dfrac{1}{2}\cos^{-1}\dfrac{a - x}{a + x}.$

17. Prove $\cos^2 \theta - \cos^2 \phi = \sin(\phi + \theta)\sin(\phi - \theta)$.

18. Prove $\tan A + \tan(A + 120°) + \tan(A - 120°) = 3\tan 3A$

19. Prove $\tan\alpha - \tan\dfrac{\alpha}{2} = \tan\dfrac{\alpha}{2}\sec\alpha.$

20. $3\tan^{-1} a = \tan^{-1}\dfrac{3a - a^3}{1 - 3a^2}.$

21. $\cos^2 3A\,(\tan^2 3A - \tan^2 A) = 8\sin^2 A \cos 2A.$

22. $1 + \cos 2(A - B)\cos 2B = \cos^2 A + \cos^2(A - 2B).$

23. $\cot^2\left(\dfrac{\pi}{4} + \dfrac{\theta}{2}\right) = \dfrac{2\csc 2\theta - \sec\theta}{2\csc 2\theta + \sec\theta}.$

TRIGONOMETRIC EQUATIONS AND IDENTITIES.

54. Identities. It was shown in Chapter I that

$$\sin^2\theta + \cos^2\theta = 1$$

is true for all values of θ, and in Chapter VI, that

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$

is true for all values of α and β. It may be shown that

$$\frac{\sin 2A}{1+\cos 2A} = \tan A$$

is true for all values of A, thus:

$$\frac{\sin 2A}{1+\cos 2A} = \frac{2\sin A\cos A}{1+2\cos^2 A - 1} \text{ (by trigonometric transformation)}$$

$$= \frac{\sin A}{\cos A} \text{ (by algebraic transformation)}$$

$$= \tan A \text{ (by trigonometric transformation).}$$

Such expressions are called *trigonometric identities. They are true for all values of the angles involved.*

55. Equations. The expression

$$2\cos^2\alpha - 3\cos\alpha + 1 = 0$$

is true for but two values of $\cos\alpha$, viz. $\cos\alpha = \frac{1}{2}$ and 1, *i.e.* the expression is true for $\alpha = 0°, 60°, 300°$, and for no other positive angles less than 360°. Such expressions are called *trigonometric equations. They are true only for particular values of the angles involved.*

56. Method of attack. The transformations necessary at any step in the proof of identities, or the solution of equations, are either *trigonometric*, or *algebraic; i.e.* in proving an identity, or solving an equation, the student must choose at each step to apply either some principles of algebra, or some trigonometric relations. If at any step no algebraic operation seems advantageous, then usually the expression

should be simplified by endeavoring to state the *different functions* involved in terms of a *single function* of the angle, or if there are *multiple angles*, to reduce all to functions of a *single angle.*

$$\text{Transformations}\begin{cases}\text{Algebraic}\\ \text{Trigonometric, to change to a}\begin{cases}\text{Single function}\\ \text{Single angle}\end{cases}\end{cases}$$

No other transformations are needed, and the student will be greatly assisted by remembering that the ready solution of a trigonometric problem consists in wisely choosing at each step between the possible algebraic and trigonometric transformations. Problems involving trigonometric functions will in general be simplified by expressing them entirely in terms of sine and cosine.

EXAMPLES.

1. Prove $$\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = 2.$$

By algebra, $$\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = \frac{\sin 3A \cos A - \cos 3A \sin A}{\sin A \cos A}$$

by trigonometry, $$= \frac{\sin(3A - A)}{\sin A \cos A} = \frac{\sin 2A}{\sin A \cos A}$$

$$= \frac{2 \sin A \cos A}{\sin A \cos A} = 2.$$

Or, by trigonometry,

$$\frac{\sin 3A}{\sin A} - \frac{\cos 3A}{\cos A} = \frac{3 \sin A - 4 \sin^3 A}{\sin A} - \frac{4 \cos^3 A - 3 \cos A}{\cos A}$$

by algebra, $$= 3 - 4 \sin^2 A - 4 \cos^2 A + 3$$

$$= 6 - 4(\sin^2 A + \cos^2 A) = 2.$$

2. Prove $$\frac{\sec 8\theta - 1}{\sec 4\theta - 1} = \frac{\tan 8\theta}{\tan 2\theta}.$$

No algebraic operation simplifies. Two trigonometric changes are needed. 1. To change the functions to a single function, sine or cosine. 2. To change the angles to a single angle, $8A$, $4A$, or $2A$.

By trigonometry and algebra,

$$\frac{\dfrac{1-\cos 8\theta}{\cos 8\theta}}{\dfrac{1-\cos 4\theta}{\cos 4\theta}}=\frac{\dfrac{\sin 8\theta}{\cos 8\theta}}{\dfrac{\sin 2\theta}{\cos 2\theta}};$$

by algebra, $$\frac{\cos 4\theta(1-\cos 8\theta)}{1-\cos 4\theta}=\frac{\sin 8\theta\cos 2\theta}{\sin 2\theta};$$

by trigonometry,

$$\frac{\cos 4\theta(1-1+2\sin^2 4\theta)}{1-1+2\sin^2 2\theta}=\frac{2\sin 4\theta\cos 4\theta\cos 2\theta}{\sin 2\theta};$$

by algebra, $$\frac{\sin 4\theta}{\sin 2\theta}=2\cos 2\theta;$$

and $$\sin 4\theta=2\sin 2\theta\cos 2\theta,$$

which is a trigonometric identity.

3. Solve $2\cos^2\theta+3\sin\theta=0$.

By trigonometry, $2(1-\sin^2\theta)+3\sin\theta=0$,

a quadratic equation in $\sin\theta$.

By algebra, $$2\sin^2\theta-3\sin\theta-2=0,$$

and $$(\sin\theta-2)(2\sin\theta+1)=0.$$

$$\therefore \sin\theta=2, \text{ or } -\tfrac{1}{2}. \quad \text{Verify.}$$

The value 2 must be rejected. Why?

$\therefore \theta=210°$, and $330°$ are the only positive values less than $360°$ that satisfy the equation.

4. Solve $\sec\theta-\tan\theta=2$.

Here $\tan\theta=-0.75$, $\therefore$ from the tables of natural functions,

$$\theta=143°\,7'\,48'', \text{ or } 323°\,7'\,48''.$$

Find $\sec\theta$, and verify.

5. Solve $2\sin\theta\sin 3\theta-\sin^2 2\theta=0$.

By trigonometry, $\cos 2\theta-\cos 4\theta-\sin^2 2\theta=0$,

also $$\cos 2\theta-\cos^2 2\theta+\sin^2 2\theta-\sin^2 2\theta=0.$$

By algebra, $$\cos 2\theta(1-\cos 2\theta)=0.$$

$$\therefore \cos 2\theta=0 \text{ or } 1,$$

and $$2\theta=90°, 270°, 0°, \text{ or } 360°,$$

whence $$\theta=45°, 135°, 0°, \text{ or } 180°. \quad \text{Verify.}$$

Or, by trigonometry,

$$2\sin\theta(3\sin\theta - 4\sin^3\theta) - 4\sin^2\theta\cos^2\theta = 0;$$

by trigonometry and algebra,

$$6\sin^2\theta - 8\sin^4\theta - 4\sin^2\theta + 4\sin^4\theta = 0;$$

by algebra, $$2\sin^2\theta - 4\sin^4\theta = 0,$$

and $$2\sin^2\theta(1 - 2\sin^2\theta) = 0.$$

$$\therefore \sin\theta = 0, \text{ or } \pm\sqrt{\tfrac{1}{2}},$$

and $$\theta = 0°, 180°, 45°, 135°, 225°, \text{ or } 315°.$$

The last two values do not appear in the first solution, because only angles less than 360° are considered, and the solution there gave values of 2θ, which in the last two cases would be 450° and 630°.

Solve: **6.** $\tan\theta = \cot\theta$. **8.** $2\cos 2\theta - 2\sin\theta = 1$.

7. $\sin^2\theta + \cos\theta = 1$. **9.** $\sin 2\theta\cos\theta = \sin\theta$.

Prove: **10.** $2\cot 2A = \cot A - \tan A$.

11. $\cos 2x + \cos 2y = 2\cos(x + y)\cos(x - y)$.

12. $(\cos\alpha + \sin\alpha)^2 = 1 + \sin 2\alpha$.

57. Simultaneous trigonometric equations.

13. Solve $$\cos(x + y) + \cos(x - y) = 2,$$ $$\sin\frac{x}{2} + \sin\frac{y}{2} = 0.$$

By trigonometry,

$$\cos x\cos y - \sin x\sin y + \cos x\cos y + \sin x\sin y = 2,$$

so that $$\cos x\cos y = 1;$$

also, $$\sqrt{\frac{1-\cos x}{2}} + \sqrt{\frac{1-\cos y}{2}} = 0,$$

and $$\therefore \cos x = \cos y.$$

Substituting, $$\cos^2 x = 1,$$

$$\cos x = \pm 1.$$

$$\therefore x = 0°, \text{ or } 180°,$$

and $$y = x = 0°, \text{ or } 180°. \quad \textbf{Verify.}$$

14. Solve for R and F.

$$W - F\sin i - R\cos i = 0,$$
$$W + F\cos i - R\sin i = 0.$$

To eliminate F,

$$W\cos i - F\sin i\cos i - R\cos^2 i = 0,$$
$$W\sin i + F\cos i\sin i - R\sin^2 i = 0.$$

Adding, $$W(\sin i + \cos i) - R(\sin^2 i + \cos^2 i) = 0.$$

$$\therefore R = W(\sin i + \cos i).$$

Substituting, $$W - F\sin i - W(\sin i + \cos i)\cos i = 0$$

$$\therefore F = \frac{W - W(\sin i + \cos i)\cos i}{\sin i}.$$

If $W = 3$ tons, and $i = 22°\,30'$, compute F and R.

$$R = 3(0.3827 + 0.9239) = 3.9198.$$

$$F = \frac{3 - 3(0.3827 + 0.9239)0.9239}{0.3827} = -1.624.$$

Solve:

15. $472\cot\theta - 263\cot\phi = 490,\quad 307\cot\theta - 379\cot\phi = 0.$

16. $\sin 2x + 1 = \cos x + 2\sin x.$

17. $\cos^2\theta + \sin\theta = 1.$

18. If $2h(\cos^2\theta - \sin^2\theta) - 2a\sin\theta\cos\theta + 2b\sin\theta\cos\theta = 0$, prove $\theta = \frac{1}{2}\tan^{-1}\dfrac{2h}{a-b}.$

Prove:

19. $\tan y = (1 + \sec y)\tan\dfrac{y}{2}.$

20. $2\cot^{-1}x = \csc^{-1}\dfrac{1+x^2}{2x}.$

21. $\sin(\phi + 45°) + \sin(\phi + 135°) = \sqrt{2}\cos\phi.$

22. $\dfrac{\cos v + \cos 3v}{\cos 3v + \cos 5v} = \dfrac{1}{2\cos 2v - \sec 2v}.$

23. $\cos 3x - \sin 3x = (\cos x + \sin x)(1 - 2\sin 2x).$

Solve:

24. $\sin 2\theta + \sin\theta = \cos 2\theta + \cos\theta.$

25. $4\cos(\theta + 60°) - \sqrt{2} = \sqrt{6} - 4\cos(\theta + 30°).$

26. $\cot 2\theta = \tan\theta - 1.$

27. $\cos\theta + \cos 2\theta + \cos 3\theta = 0.$

28. $\sin 2x + \sqrt{3}\cos 2x = 1$.

29. $3\tan^2 p + 8\cos^2 p = 7$.

30. Determine for what relative values of P and W the following equation is true:

$$\cos^2\frac{y}{2} - \frac{P}{W}\cos\frac{y}{2} - \frac{1}{2} = 0.$$

31. Compute N from the equation $N + \frac{W}{3}\cos\alpha - \frac{W}{3}\sin\alpha - W\cos\alpha = 0$, when $W = 2000$ pounds and α satisfies the equation $2\sin\alpha = 1 + \cos\alpha$.

32. $\sin\theta - \tan\phi(\cos\theta + \sin\theta) = \cos\theta$, $\sin\theta - \tan\phi\cos\theta = 1$.

Prove:

33. $\cot(t + 15°) - \tan(t - 15°) = \dfrac{4\cos 2t}{2\sin 2t + 1}$.

34. $\sin^{-1}\frac{3}{5} - \sin^{-1}\frac{5}{13} = \sin^{-1}\frac{16}{65}$.

35. $\tan\left(\dfrac{\pi}{4} + \dfrac{\omega}{2}\right) = \sqrt{\dfrac{1 + \sin\omega}{1 - \sin\omega}}$.

36. $2\sin^{-1}\frac{1}{2} = \cos^{-1}\frac{1}{2}$.

37. If $\sin A$ is a geometric mean between $\sin B$ and $\cos B$, prove $\cos 2A = 2\sin(45 - B)\cos(45 + B)$.

38. Prove $\sin(\alpha + \beta + \gamma) = \sin\alpha\cos\beta\cos\gamma + \cos\alpha\sin\beta\cos\gamma$
$+ \cos\alpha\cos\beta\sin\gamma - \sin\alpha\sin\beta\sin\gamma$.

Also find $\cos(\alpha + \beta + \gamma)$.

39. Prove $\tan(\alpha + \beta + \gamma) = \dfrac{\tan\alpha + \tan\beta + \tan\gamma - \tan\alpha\tan\beta\tan\gamma}{1 - \tan\alpha\tan\beta - \tan\beta\tan\gamma - \tan\gamma\tan\alpha}$.

If α, β, and γ are angles of a triangle, prove

40. $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha\tan\beta\tan\gamma$.

41. $\cot\dfrac{\alpha}{2} + \cot\dfrac{\beta}{2} + \cot\dfrac{\gamma}{2} = \cot\dfrac{\alpha}{2}\cot\dfrac{\beta}{2}\cot\dfrac{\gamma}{2}$.

If $\alpha + \beta + \gamma = 90°$, prove

42. $\tan\alpha\tan\beta + \tan\beta\tan\gamma + \tan\gamma\tan\alpha = 1$.

Prove:

43. $\sin n\alpha = 2\sin(n - 1)\alpha\cos\alpha - \sin(n - 2)\alpha$.

44. $\cos n\alpha = 2\cos(n - 1)\alpha\cos\alpha - \cos(n - 2)\alpha$.

45. $\tan n\alpha = \dfrac{\tan(n - 1)\alpha + \tan\alpha}{1 - \tan(n - 1)\alpha\tan\alpha}$.

CHAPTER VII.

TRIANGLES.

58. In geometry it has been shown that a triangle is determined, except in the ambiguous case, if there are given any three independent parts, as follows:

I. Two angles and a side.

II. Two sides and an angle,

(*a*) the angle being included by the given sides,

(*b*) the angle being opposite one of the given sides (ambiguous case).

III. Three sides.

The angles of a triangle are not three *independent* parts, since they are connected by the relation $A + B + C = 180°$.

The three angles of a triangle will be designated A, B, C, the sides opposite, a, b, c.

But the principles of geometry do not enable us to *compute* the unknown parts. This is accomplished by the following laws of trigonometry:

I. *Law of Sines*, $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

II. *Law of Tangents*, $$\frac{\tan \frac{1}{2}(A - B)}{\tan \frac{1}{2}(A + B)} = \frac{a - b}{a + b}, \text{ etc.}$$

III. *Law of Cosines*, $$\cos A = \frac{b^2 + c^2 - a^2}{2bc}, \text{ etc.}$$

59. Law of Sines. *In any triangle the sides are proportional to the sines of the angles opposite.*

Let ABC be any triangle, p the perpendicular from B on b. In I (Fig. 34), C is an *acute*, in II, an *obtuse*, in III,

a *right* angle. The demonstration applies to each triangle, but in II, $\sin ACB = \sin DCB$ (why?); in III, $\sin C = 1$ (why?).

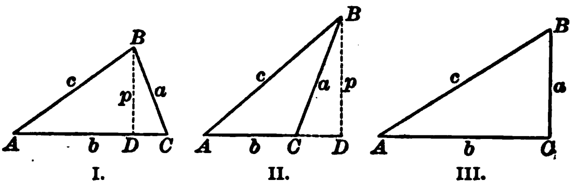


FIG. 34.

Now $\sin A = \frac{p}{c}$, $\therefore\ p = c \sin A.$

$\sin C = \frac{p}{a}$, $\therefore\ p = a \sin C.$

Equating values of p, $c \sin A = a \sin C$,

or, $$\frac{\sin A}{a} = \frac{\sin C}{c}.$$

By dropping a perpendicular from A, or C, on a or c, show that

$$\frac{\sin B}{b} = \frac{\sin C}{c}, \text{ or } \frac{\sin A}{a} = \frac{\sin B}{b},$$

whence $$\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}.$$

60. Law of Tangents. *The tangent of half the difference of two angles of a triangle is to the tangent of half their sum, as the difference of the sides opposite is to their sum.*

By Art. 59, $$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

By composition and division,

$$\frac{a-b}{a+b} = \frac{\sin A - \sin B}{\sin A + \sin B} = \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}$$

$$= \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)};$$

or, $$\frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)} = \frac{a-b}{a+b}.$$

61. Law of Cosines. *The cosine of any angle of a triangle is equal to the quotient of the sum of the squares of the adjacent sides less the square of the opposite side, divided by twice the product of the adjacent sides.*

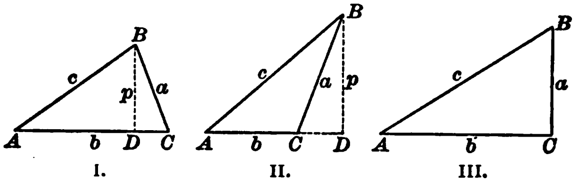


FIG. 34.

In each figure $$a^2 = p^2 + DC^2$$

$$= c^2 - AD^2 + (b - AD)^2$$

(in Fig. 34, II, DC is negative; in III, zero)

$$= c^2 - AD^2 + b^2 - 2b \cdot AD + AD^2$$

$$= b^2 + c^2 - 2b \cdot AD.$$

But

$AD = c\cos A$, $\quad \therefore\ a^2 = b^2 + c^2 - 2bc\cos A;$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

Prove that $$\cos B = \frac{a^2 + c^2 - b^2}{2ac},$$

and $$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

62. Though these formulæ may be used for the solution of the triangle, they are not adapted to the use of logarithms (why?). Hence we derive the following:

Since $\cos A = 2\cos^2\frac{A}{2} - 1 = 1 - 2\sin^2\frac{A}{2},$

we have

$$2\cos^2\frac{A}{2} = 1 + \cos A, \text{ and } 2\sin^2\frac{A}{2} = 1 - \cos A.$$

From the latter

$$2\sin^2\frac{A}{2}=1-\frac{b^2+c^2-a^2}{2bc}=\frac{2bc-b^2-c^2+a^2}{2bc}$$

$$=\frac{a^2-(b-c)^2}{2bc}=\frac{(a-b+c)(a+b-c)}{2bc}.$$

Let $a+b+c=2s$, then $a+b-c=a+b+c-2c=2s-2c$;

i.e. $$a+b-c=2(s-c).$$

In like manner, $$a-b+c=2(s-b).$$

$$-a+b+c=2(s-a).$$

Substituting, $$2\sin^2\frac{A}{2}=\frac{2(s-b)\cdot 2(s-c)}{2bc}.$$

$$\therefore\ \sin\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{bc}}.$$

Show that $$\sin\frac{B}{2}=?$$

also $$\sin\frac{C}{2}=?$$

From $$2\cos^2\frac{A}{2}=1+\cos A,$$

show that $$\cos\frac{A}{2}=\sqrt{\frac{s(s-a)}{bc}},$$

also $$\cos\frac{B}{2}=?$$

and $$\cos\frac{C}{2}=?$$

Also derive the formulæ

$$\tan\frac{A}{2}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}},$$

$$\tan\frac{B}{2}=?$$

$$\tan\frac{C}{2}=?$$

63. Area of the triangle. In the figures of Art. 59 the area of the triangle $ABC = \Delta = \frac{1}{2}pb$.

But $p = c \sin A$. $\therefore \Delta = \frac{1}{2} bc \sin A$. (i)

Again, by law of sines, $b = \dfrac{c \sin B}{\sin C}$.

Substituting, $$\Delta = \frac{c^2 \sin A \sin B}{2 \sin C}$$

$$= \frac{c^2 \sin A \sin B}{2 \sin(A+B)} \text{ (why?)}. \qquad \text{(ii)}$$

Finally, since $\sin A = 2 \sin \dfrac{A}{2} \cos \dfrac{A}{2}$, we have from (i)

$$\Delta = \tfrac{1}{2} bc \cdot 2 \sin \frac{A}{2} \cos \frac{A}{2} = bc \sqrt{\frac{s(s-a)(s-b)(s-c)}{bc \cdot bc}}$$

or $$\Delta = \sqrt{s(s-a)(s-b)(s-c)}. \qquad \text{(iii)}$$

Find Δ; (1) Given $a = 10$, $b = 12$, $C = 45°$.
(2) Given $a = 4$, $b = 5$, $c = 6$.
(3) Given $a = 2$, $B = 45°$, $C = 60°$.

SOLUTION OF TRIANGLES.

64. For the solution of triangles we have the following formulæ, which should be carefully memorized:

I. $\dfrac{\sin A}{a} = \dfrac{\sin B}{b} = \dfrac{\sin C}{c}$.

II. $\tan \frac{1}{2}(A - B) = \dfrac{a-b}{a+b} \tan \frac{1}{2}(A + B)$.

III. $\sin \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{bc}}$, or $\cos \dfrac{A}{2} = \sqrt{\dfrac{s(s-a)}{bc}}$,

or $\tan \dfrac{A}{2} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}}$.

IV. $\Delta = \frac{1}{2} bc \sin A = \dfrac{c^2 \sin A \sin B}{2 \sin(A + B)} = \sqrt{s(s-a)(s-b)(s-c)}$.

Which of the above formulæ shall be used in the solution of a given triangle must be determined by examining the parts known, as will appear in Art. 69. It is always possible to express each of the unknown parts in terms of three known parts.

In solving triangles such as Case I, Art. 58, the law of sines applies; for, if the given side is not opposite either given angle, the third angle of the triangle is found from the relation $A + B + C = 180°$, and then three of the four quantities in $\frac{\sin A}{a} = \frac{\sin B}{b}$ being known, the solution gives the fourth.

In Case II (b) the law of sines applies, but in II (a) two only of the four quantities in $\frac{\sin A}{a} = \frac{\sin B}{b}$ are known. Therefore, we resort to the formula

$$\tan \tfrac{1}{2}(A - B) = \frac{a-b}{a+b} \tan \tfrac{1}{2}(A + B),$$

in which all the factors of the second member are known.

In Case III, $\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$ is clearly applicable, and is *preferred* to the formulæ for $\sin\frac{A}{2}$ and $\cos\frac{A}{2}$; for, first, it is more accurate since tangent varies in magnitude from 0 to ∞, while sine and cosine lie between 0 and 1. (See Art. 27, 5.)

Let the student satisfy himself on this point by finding, correct to seconds, the angle whose logarithmic sine is 9.99992, and whose logarithmic tangent is 1.71668. Does the first determine the angle? Does the second?

And, second, it is more convenient, since in the complete solution of the triangle by $\sin\frac{A}{2}$ *six* logarithms must be taken from the table, by $\cos\frac{A}{2}$ *seven*, and by $\tan\frac{A}{2}$ but *four*.

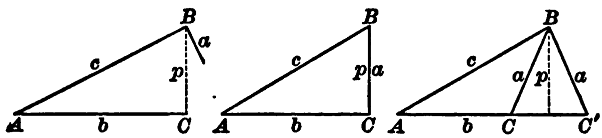
The right triangle may be solved as a special case by the law of sines, since $\sin C = 1$.

65. Ambiguous case. In geometry it was proved that a triangle having two sides and an angle opposite one of them of given magnitude is not always determined. The marks of the undetermined or ambiguous triangle are:

1. *The parts given are two sides and an angle opposite one.*
2. *The given angle is acute.*
3. *The side opposite this angle is less than the other given side.*

When these marks are all present, the number of solutions must be tested in one of two ways:

(*a*) From the figure it is apparent that there will be *no solution* when the side opposite is less than the perpendicular p; *one solution* when side a equals p; and *two solutions* when a is greater than p.



No Solution. *One Solution.* *Two Solutions.*

FIG. 35.

And since $\sin A = \frac{p}{c}$, it follows that there will be *no solution, one solution, two solutions*, according as $\sin A \gtreqless \frac{a}{c}$.

(*b*) A good test is found in solving by means of logarithms; and there will be *no solutions, one solution, two solutions*, according as log sin C proves to be *impossible, zero, possible, i.e.* as log sin C is positive, zero, or negative. This results from the fact that sine cannot be greater than unity, whence log sine must have a negative characteristic, or be zero.

66. In computations *time* and *accuracy* assume more than usual importance. *Time* will be saved by an orderly arrangement of the formulæ for the complete solution, before opening the book of logarithms, thus:

Given A, B, a. Solve completely.

$$C = 180^\circ - (A+B),\quad b = \frac{a \sin B}{\sin A},\quad c = \frac{a \sin C}{\sin A},\quad \Delta = \tfrac{1}{2} ab \sin C.$$

$$\begin{array}{rl}
 & 180^\circ \\
A + B = & \underline{\hspace{5em}} \\
\therefore C = &
\end{array}
\qquad
\begin{array}{rl}
\log a = & \\
\log \sin B = & \\
\operatorname{colog} \sin A = & \underline{\hspace{5em}} \\
\log b = & \\
\therefore b = &
\end{array}
\qquad
\begin{array}{rl}
\log a = & \\
\log \sin C = & \\
\operatorname{colog} \sin A = & \underline{\hspace{5em}} \\
\log c = & \\
\therefore c = &
\end{array}$$

Check:

$$\begin{array}{rl}
\log a = & \\
\log b = & \\
\log \sin C = & \\
\operatorname{colog} 2 = & \underline{\hspace{5em}} \\
\log \Delta = & \\
\therefore \Delta = &
\end{array}
\qquad
\begin{array}{rl}
\log (s - b) = & \\
\log (s - c) = & \\
\operatorname{colog} s = & \\
\operatorname{colog} (s - a) = & \underline{\hspace{5em}} \\
2\big) & \underline{\hspace{5em}} \\
\log \tan \frac{A}{2} = & \\
\therefore A = &
\end{array}$$

67. *Accuracy* must be secured by checks on the work at every step; *e.g.* in adding columns of logarithms, first add up, and then check by adding down. Too much care cannot be given to verification in the simple operations of addition, subtraction, multiplication, and division. A final check should be made by using other formulæ involving the parts in a different way, as in the check above. As far as possible the parts originally given should be used throughout in the solution, so that an error in computing one part may not affect later computations.

68. The formulæ should always be *solved for the unknown part before using*, and it should be noted whether the solution gives one value, or more than one, for each part; *e.g.* the same value of $\sin B$ belongs to two supplementary angles, one or both of which may be possible, as in the ambiguous case.

69. Write formulæ for the complete solution of the following triangles, showing whether you find no solution, one solution, two or more solutions, in each case, with reasons for your conclusion:

	a	b	c	A	B	C
1.				81° 26′ 28″	44° 11′ 20″	54° 22′ 12″
2.		78.54		63° 18′ 20″		41° 30′ 18″
3.		135.82	26.89	53° 28′ 30″		
4.	0.75	0.85	0.95			
5.	243		562			36° 15′ 40″
6.		38.75	25.92			63° 50′ 10″
7.	0.058			78° 15′	83° 46′	
8.	2986		1493			30°
9.		48	50		26° 15′	

MODEL SOLUTIONS.

1. Given $a = 0.785$, $b = 0.85$, $c = 0.633$. Solve completely.

$$\tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}},\quad \tan\frac{B}{2} = \sqrt{\frac{(s-a)(s-c)}{s(s-b)}},\quad \tan\frac{C}{2} = \sqrt{\frac{(s-a)(s-b)}{s(s-c)}}.$$

Check: $A + B + C = 180°$. $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$.

$$\begin{aligned} a &= 0.735 \\ b &= 0.85 \\ c &= \underline{0.633} \\ 2)&\underline{2.268} \\ s &= 1.134 \\ s - a &= 0.349 \\ s - b &= 0.284 \\ s - c &= 0.501 \end{aligned}$$

$$\begin{aligned} \log(s-b) &= 9.45332 \\ \log(s-c) &= 9.69984 \\ \operatorname{colog} s &= 9.94539 \\ \operatorname{colog}(s-a) &= \underline{0.45717} \\ 2)&\underline{19.55572} \\ \log\tan\tfrac{1}{2}A &= 9.77786 \\ \tfrac{1}{2}A &= 30°\,56'\,49'' \\ A &= 61°\,53'\,38'' \end{aligned}$$

$$\begin{aligned} \log(s-a) &= 9.54283 \\ \log(s-c) &= 9.69984 \\ \operatorname{colog} s &= 9.94539 \\ \operatorname{colog}(s-b) &= \underline{0.54668} \\ 2)&\underline{19.73474} \\ \log\tan\tfrac{1}{2}B &= 9.86737 \\ \tfrac{1}{2}B &= 36°\,23'\,2'' \\ B &= 72°\,46'\,4'' \end{aligned}$$

Check:

$$\begin{aligned} A &= 61°\,53'\,38'' \\ B &= 72°\,46'\,\ 4'' \\ C &= \underline{45°\,20'\,20''} \\ & 180°\ \ 0'\ \ 2'' \end{aligned}$$

$$\begin{aligned} \log(s-a) &= 9.54283 \\ \log(s-b) &= 9.45332 \\ \operatorname{colog} s &= 9.94539 \\ \operatorname{colog}(s-c) &= \underline{0.30016} \\ 2)&\underline{19.24170} \\ \log\tan\tfrac{1}{2}C &= 9.62085 \\ \tfrac{1}{2}C &= 22°\,40'\,10'' \\ C &= 45°\,20'\,20'' \end{aligned}$$

$$\begin{aligned} \log s &= 0.05461 \\ \log(s-a) &= 9.54283 \\ \log(s-b) &= 9.45332 \\ \log(s-c) &= \underline{9.69984} \\ 2)&\underline{18.75060} \\ \log\Delta &= 9.37530 \\ \Delta &= 0.2373 \end{aligned}$$

Solve: (1) Given $a = 30$, $b = 40$, $c = 50$.

(2) Given $a = 2159$, $b = 1431.6$, $c = 914.8$.

(3) Given $a = 78.54$, $b = 32.56$, $c = 48.9$.

2. Given $A = 57°\,23'\,12''$, $C = 68°\,15'\,30''$, $c = 832.56$. Solve completely.

$$a = \frac{c \sin A}{\sin C},\quad b = \frac{c \sin B}{\sin C},\quad \Delta = \tfrac{1}{2} bc \sin A.$$

$$\begin{aligned} B &= 180° - (A + C) \\ &= 54°\,21'\,18''. \end{aligned} \qquad \textit{Check:}\ \tan \tfrac{1}{2} A = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}.$$

$$\begin{array}{rl} \log c =& 2.92042 \\ \log \sin A =& 9.92548 \\ \text{colog} \sin C =& \underline{0.03204} \\ \log a =& 2.87794 \\ a =& 754.98 \end{array} \qquad \begin{array}{rl} \log c =& 2.92042 \\ \log \sin B =& 9.90990 \\ \text{colog} \sin C =& \underline{0.03204} \\ \log b =& 2.86236 \\ b =& 728.38 \end{array} \qquad \begin{array}{rl} \log b =& 2.86236 \\ \log c =& 2.92042 \\ \log \sin A =& \underline{9.92548} \\ \log 2\Delta =& 5.70826 \\ \Delta =& \dfrac{510811}{2} = 255405.5 \end{array}$$

$$\begin{array}{rl} \textit{Check:}\ a =& 754.98 \\ b =& 728.38 \\ c =& \underline{832.56} \\ 2)& \underline{2315.92} \\ s =& 1157.96 \end{array} \qquad \begin{array}{rl} s - a =& 402.98 \\ s - b =& 429.58 \\ s - c =& 325.40 \end{array} \qquad \begin{array}{rl} \log(s-b) =& 2.63304 \\ \log(s-c) =& 2.51242 \\ \text{colog}\, s =& 6.93634 \\ \text{colog}(s-a) =& \underline{7.39471} \\ 2)& \underline{19.47651} \\ \log \tan \tfrac{1}{2} A =& 9.73826 \\ \tfrac{1}{2} A =& 28°\,41'\,38' \\ A =& 57°\,23'\,16' \end{array}$$

Solve:

(1) Given $a = 215.73$, $B = 92°\,15'$, $C = 28°\,14'$.
(2) Given $b = 0.827$, $A = 78°\,14'\,20''$, $B = 63°\,42'\,30''$.
(3) Given $b = 7.54$, $c = 6.93$, $B = 54°\,28'\,40''$.

3. Given $a = 25.384$, $c = 52.925$, $B = 28°\,32'\,20''$. Solve completely.

(Why not use the same formulæ as in Example 1, or 2?)

$$\tan \frac{C-A}{2} = \frac{c-a}{c+a} \tan \frac{C+A}{2},\quad b = \frac{c \sin B}{\sin C},\quad \Delta = \tfrac{1}{2} ac \sin B.$$

$$\begin{aligned} 180° - B = C + A &= 151°\,27'\,40''. \\ \therefore \tfrac{1}{2}(C + A) &= 75°\,43'\,50''. \end{aligned} \qquad \textit{Check:}\ b = \frac{a \sin B}{\sin A}.$$

$$\begin{array}{rl} c =& 52.925 \\ a =& \underline{25.384} \\ c+a =& 78.309 \\ c-a =& 27.541 \end{array} \qquad \begin{array}{rl} \log(c-a) =& 1.43998 \\ \text{colog}(c+a) =& 8.10619 \\ \log \tan \tfrac{1}{2}(C+A) =& \underline{0.59460} \\ \log \tan \tfrac{1}{2}(C-A) =& 0.14077 \end{array} \qquad \begin{array}{rl} \therefore \tfrac{1}{2}(C-A) =& 54°\,7'38'' \\ \tfrac{1}{2}(C+A) =& \underline{75°43'50''} \\ \text{adding, } C =& 129°51'28'' \\ \text{subtracting, } A =& 21°36'12'' \end{array}$$

$$\begin{array}{rl} \log c =& 1.72366 \\ \log \sin B =& 9.67921 \\ \text{colog} \sin C =& \underline{0.11484} \\ \log b =& 1.51771 \\ b =& 32.939 \end{array} \qquad \begin{array}{rl} \textit{Check:}\ \log a =& 1.40456 \\ \log \sin B =& 9.67921 \\ \text{colog} \sin A =& \underline{0.43395} \\ \log b =& 1.51772 \end{array} \qquad \begin{array}{rl} \log a =& 1.40456 \\ \log c =& 1.72366 \\ \log \sin B =& \underline{9.67921} \\ \log 2\Delta =& 2.80743 \\ \Delta =& \dfrac{641.84}{2} = 320.92 \end{array}$$

Solve: (1) Given $a = 0.325$, $c = 0.426$, $B = 48° 50' 10''$.

(2) Given $b = 4291$, $c = 3194$, $A = 73° 24' 50''$.

(3) Given $b = 5.38$, $c = 12.45$, $A = 62° 14' 40''$.

4. *Ambiguous cases.* Since the required angle is found in terms of its sine, and since $\sin \alpha = \sin (180° - \alpha)$, it follows that there may be two values of α, one in the first, and the other in the second quadrant, their sum being 180°. In the following examples the student should note that all the marks of the ambiguous case are present. The solutions will show the treatment of the ambiguous triangle having no solution, one solution, two solutions.

(*a*) Given $b = 70$, $c = 40$, $C = 47° 32' 10''$. Solve. Why ambiguous?

$$\sin B = \frac{b \sin C}{c}.$$

$$\begin{aligned} \log b &= 1.84510 \\ \log \sin C &= 9.86788 \\ \operatorname{colog} c &= \underline{8.39794} \\ \log \sin B &= 0.11092 \end{aligned}$$

$\therefore$ B is impossible, and there is no solution. Why? Show the same by $\sin C > \frac{c}{b}$.

(*b*) Given $a = 1.5$, $c = 1.7$, $A = 61° 55' 38''$. Solve.

$$\sin C = \frac{c \sin A}{a}.$$

$$\begin{aligned} \log c &= 0.23045 \\ \log \sin A &= 9.94564 \\ \operatorname{colog} a &= \underline{9.82391} \\ \log \sin C &= 0.00000 \\ C &= 90° \end{aligned}$$

and there is one solution. Why? Show the same by $\sin A = \frac{a}{c}$. Solve for the remaining parts and check the work.

(*c*) Given $a = 0.235$, $b = 0.189$, $B = 36° \; 28' \; 20''$. Solve.

$$\sin A = \frac{a \sin B}{b}, \qquad c = \frac{b \sin C}{\sin B}.$$

$$\begin{array}{rlrll}
\log a = & 9.37107 & \log b = & 9.27646 & \; 9.27646 \\
\log \sin B = & 9.77411 & \log \sin C = & 9.99772 & \text{or } 9.28774 \\
\text{colog } b = & \underline{0.72354} & \text{colog} \sin B = & \underline{0.22589} & \; \underline{0.22589} \\
\log \sin A = & 9.86872 & \log c = & 9.50007 & \text{or } 8.79009 \\
A = & 47° \; 39' \; 25'' & c = & 0.31628 & \text{or } 0.06167 \\
& \text{or } 132° \; 20' \; 35''. & & &
\end{array}$$

$\therefore \; C = 95° \; 52' \; 15''$ or $11° \; 11' \; 5''$.

Solve for Δ, and check. Show the same by $\sin B < \frac{b}{a}$.

Solve :

(1) Given $b = 216.4$, $c = 593.2$, $B = 98° \; 15'$.

(2) Given $a = 22$, $b = 75$, $B = 32° \; 20'$.

(3) Given $a = 0.353$, $c = 0.295$, $A = 46° \; 15' \; 20''$.

(4) Given $a = 293.445$, $b = 450$, $A = 40° \; 42'$.

(5) Given $b = 531.03$, $c = 629.20$, $B = 34° \; 28' \; 16''$.

. Solve completely, given :

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	a	b	c	A	B	C
1.	50	60				78° 27′ 47″
2.		10	11			93° 35′
3.	4	5	6			
4.			10	109° 28′ 16″	38° 56′ 54″	
5.	40	51		49° 28′ 32″		
6.	352.25	513.27	482.68			
7.	0.573	0.394		112° 4′		
8.	107.087			56° 15′	48° 35′	
9.			$\sqrt{2}$	117°	45°	
10.	197.63	246.35		34° 27′		
11.	4090	3850	3811			
12.	3795				73° 15′ 15″	42° 18′ 30″
13.		234.7	185.4	84° 36′		
14.		26.234	22.6925		49° 8′ 24″	
15.	273	136		72° 25′ 13″		

APPLICATIONS.

70. Measurements of heights and distances often lead to the solution of oblique triangles. With this exception, the methods of Chapter V apply, as will be illustrated in the following problems.

The *bearing* of a line is the angle it makes with a north and south line, as determined by the magnetic needle of the mariner's compass. If the bearing does not correspond to any of the points of the compass, it is usual to express it thus: N. 40° W., meaning that the line bears from N. 40° toward W.

EXAMPLES.

1. When the altitude of the sun is 48°, a pole standing on a slope inclined to the horizon at an angle of 15° casts a shadow directly down the slope 44.3 ft. How high is the pole?

2. A tree standing on a mountain side rising at an angle of 18° 30′ breaks 32 ft. from the foot. The top strikes down the slope of the mountain 28 ft. from the foot of the tree. Find the height of the tree.

3. From one corner of a triangular lot the other corners are found to be 120 ft. E. by N., and 150 ft. S. by W. Find the area of the lot, and the length of the fence required to enclose it.

4. A surveyor observed two inaccessible headlands, A and B. A was W. by N. and B, N.E. He went 20 miles N., when they were S.W. and S. by E. How far was A from B?

5. The bearings of two objects from a ship were N. by W. and N.E. by N. After sailing E. 11 miles, they were in the same line W.N.W. Find the distance between them.

6. From the top and bottom of a vertical column the elevation angles of the summit of a tower 225 ft. high and standing on the same horizontal plane are 45° and 55°. Find the height of the column.

7. An observer in a balloon 1 mile high observes the depression angle of an object on the ground to be 35° 20′. After ascending vertically and uniformly for 10 mins., he observes the depression angle of the same object to be 55° 40′. Find the rate of ascent of the balloon in miles per hour.

8. A statue 10 ft. high standing on a column subtends, at a point 100 ft. from the base of the column and in the same horizontal plane, the same angle as that subtended by a man 6 ft. high, standing at the foot of the column. Find the height of the column.

9. From a balloon at an elevation of 4 miles the dip of the horizon is 2° 38′ 40″. Required the earth's radius.

10. Two ships sail from Boston, one S.E. 50 miles, the other N.E. by E. 60 miles. Find the bearing and distance of the second ship from the first.

11. The sides of a valley are two parallel ridges sloping at an angle of 30°. A man walks 200 yds. up one slope and observes the angle of elevation of the other ridge to be 15°. Show that the height of the observed ridge is 273.2 yds.

12. To determine the height of a mountain, a north and south base line 1000 yds. long is measured; from one end of the base line the summit bears E. 10° N., and is at an altitude of 13° 14′. From the other end it bears E. 46° 30′ N. Find the height of the mountain.

13. The shadow of a cloud at noon is cast on a spot 1600 ft. due west of an observer. At the same instant he finds that the cloud is at an elevation of 23° in a direction W. 14° S. Find the height of the cloud and the altitude of the sun.

14. From the base of a mountain the elevation of its summit is 54° 20′. From a point 3000 ft. toward the summit up a plane rising at an angle of 25° 30′ the elevation angle is 68° 42′. Find the height of the mountain.

15. From two observations on the same meridian, and 92° 14′ apart, the zenith angles of the moon are observed to be 44° 54′ 21″ and 48° 42′ 57″. Calling the earth's radius 3956.2 miles, find the distance to the moon.

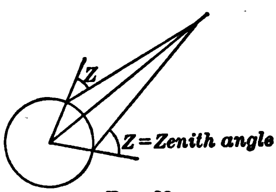

FIG. 36.

16. The distances from a point to three objects are 1130, 1850, 1456, and the angles subtended by the distances between the three objects are respectively 102° 10′, 142°, and 115° 50′. Find the distances between the three objects.

17. From a ship A running N.E. 6 mi. an hour direct to a port distant 35 miles, another ship B is seen steering toward the same port, its bearing from A being E.S.E., and distance 12 miles. After keeping on their courses $1\frac{1}{2}$ hrs., B is seen to bear from A due E. Find B's course and rate of sailing.

18. From the mast of a ship 64 ft. high the light of a lighthouse is just visible when 30 miles distant. Find the height of the lighthouse, the earth's radius being 3956.2 miles.

19. From a ship two lighthouses are observed due N.E. After sailing 20 miles E. by S., the lighthouses bear N.N.W. and N. by E. Find the distance between the lighthouses.

20. A lighthouse is seen N. 20° E. from a vessel sailing S. 25° E. A mile further on it appears due N. Determine its distance at the last observation.

EXAMPLES FOR REVIEW.

In connection with each problem the student should review all principles involved. The following list of problems will then furnish a thorough review of the book. In solving equations, find all values of the unknown angle less than 360° that satisfy the equation.

1. If $\tan\alpha = \frac{1}{7}$, $\tan\beta = \frac{1}{2}$, show that $\tan(\beta - 2\alpha) = \frac{2}{11}$.

2. Prove $\tan\alpha + \cot\alpha = 2\csc 2\alpha$.

3. From the identities $\sin^2\frac{A}{2} + \cos^2\frac{A}{2} = 1$, and $2\sin\frac{A}{2}\cos\frac{A}{2} = \sin A$,

prove $$2\sin\frac{A}{2} = \pm\sqrt{1+\sin A} \pm \sqrt{1-\sin A},$$

and $$2\cos\frac{A}{2} = \pm\sqrt{1+\sin A} \mp \sqrt{1-\sin A}.$$

4. Remove the ambiguous signs in Ex. 3 when A is in turn an angle of each quadrant.

5. A wall 20 feet high bears S. 59° 5′ E.; find the width of its shadow on a horizontal plane when the sun is due S. and at an altitude of 60°.

6. Solve $\sin x + \sin 2x + \sin 3x = 1 + \cos x + \cos 2x$.

7. Prove $\tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3} = \frac{\pi}{4}$.

8. If $A = 60°$, $B = 45°$, $C = 30°$, evaluate

$$\frac{\tan A + \tan B + \tan C}{\tan A\tan B + \tan B\tan C + \tan C\tan A}.$$

9. Prove $\dfrac{\cos(A+B)\cos C}{\cos(A+C)\cos B} = \dfrac{1-\tan A\tan B}{1-\tan A\tan C}.$

10. Solve completely the triangle whose known parts are $b = 2.35$, $c = 1.96$, $C = 38° 45'.4$.

11. Find the functions of 18°, 36°, 54°, 72°.

Let $x = 18°$. Then $2x = 36°$, $3x = 54°$, and $2x + 3x = 90°$.

12. If $\cot\alpha = \frac{p}{q}$, find the value of

$$\sin\alpha + \cos\alpha + \tan\alpha + \cot\alpha + \sec\alpha + \csc\alpha.$$

13. Prove $\dfrac{\sin 3\alpha \sin 2\beta - \sin 3\beta \sin 2\alpha}{\sin 2\alpha \sin \beta - \sin 2\beta \sin \alpha} = 1 + 4\cos\alpha\cos\beta.$

14. From a ship sailing due N., two lighthouses bear N.E. and N.N.E., respectively; after sailing 20 miles they are observed to bear due E. Find the distance between the lighthouses.

15. Solve $1 - 2\sin x = \sin 3x.$

16. Prove $\sin^{-1}\sqrt{\dfrac{a}{a+b}} = \tan^{-1}\sqrt{\dfrac{a}{b}}.$

17. If $\cos\theta - \sin\theta = \sqrt{2}\sin\theta$, then $\cos\theta + \sin\theta = \sqrt{2}\cos\theta.$

18. Solve completely the triangle ABC, given $a = 0.256$, $b = 0.387$, $C = 102°\,20'.5.$

19. Prove $\tan(30° + \alpha)\tan(30° - \alpha) = \dfrac{2\cos 2\alpha - 1}{2\cos 2\alpha + 1}.$

20. Solve $\tan(45° - \theta) + \tan(45° + \theta) = 4.$

21. Prove $\sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = \sin^2\alpha - \sin^2\beta.$

22. Prove $\cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta = \cos^2\alpha - \sin^2\beta.$

23. A man standing due S. of a water tower 150 feet high finds its elevation to be 72° 30′; he walks due W. to A street, where the elevation is 44° 50′; proceeding in the same direction one block to B street, he finds the elevation to be 22° 30′. What is the length of the block between A and B streets?

24. Prove $\tan^{-1}\dfrac{1}{3} + \tan^{-1}\dfrac{1}{5} + \tan^{-1}\dfrac{1}{7} + \tan^{-1}\dfrac{1}{8} = \dfrac{\pi}{4}.$

25. If $P = 60°$, $Q = 45°$, $R = 30°$, evaluate

$$\frac{\sin P\cos Q + \tan P\cos Q}{\sin P\cos P + \cot P\cot R}.$$

26. If $\cos(90° + \alpha) = -\frac{3}{4}$, evaluate $3\cos 2\alpha + 4\sin 2\alpha.$

27. If $\sin B + \sin C = m$, $\cos B + \cos C = n$, show that $\tan\dfrac{B+C}{2} = \dfrac{m}{n}.$

28. Show that $\sin 2\beta$ can never be greater than $2\sin\beta.$

29. Prove $\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{5}{13} = \tan^{-1}\frac{56}{33}.$

30. Solve $\cot^{-1}x + \sin^{-1}\dfrac{1}{5}\sqrt{5} = \dfrac{\pi}{4}.$

31. Solve $\sin^{-1}x + \sin^{-1}(1 - x) = \cos^{-1}x.$

32. A man standing between two towers, 200 feet from the base of the higher, which is 90 feet high, observes their elevations to be the same; 70 feet nearer the shorter tower he finds the elevation of one is twice that of the other. Find the height of the shorter tower, and his original distance from it.

33. Solve $\cos 3\beta + 8\cos^3\beta = 0$.

34. Solve $\cot m - \tan(180^\circ + m) = \sec m + \sec(90^\circ - m)$.

35. Solve $\dfrac{1 - \tan t}{1 + \tan t} = 2\cos 2t$.

36. Prove $\cot A + \cot B = \dfrac{\sin(A + B)}{\sin A \sin B}$.

37. Prove $\cot P - \cot Q = -\dfrac{\sin(P - Q)}{\sin P \sin Q}$.

38. In the triangle ABC prove

$$a = b\cos C + c\cos B,$$
$$b = c\cos A + a\cos C,$$
$$c = a\cos B + b\cos A.$$

39. Solve completely the triangle, given

$$a = 927.56,\ b = 648.25,\ c = 738.42.$$

40. Prove $\cos^2\alpha - \sin(30^\circ + \alpha)\sin(30^\circ - \alpha) = \frac{3}{4}$.

41. Prove $\tan 3x \tan x = \dfrac{\cos 2x - \cos 4x}{\cos 2x + \cos 4x}$.

42. Simplify $\cos(270^\circ + \alpha) + \sin(180^\circ + \alpha) + \cos(90^\circ + \alpha)$.

43. Simplify $\tan(270^\circ - \theta) - \tan(90^\circ + \theta) + \tan(270^\circ + \theta)$.

44. Solve $\cos 3\phi - \cos 2\phi + \cos\phi = 0$.

45. Solve $\cos A + \cos 3A + \cos 5A + \cos 7A = 0$.

46. The topmast of a yacht from a point on the deck subtends the same angle α, that the part below it does. Show that if the topmast be a feet high, the length of the part below it is $a\cos 2\alpha$.

47. A horizontal line AB is measured 400 yards long. From a point in AB a balloon ascends vertically till its elevation angles at A and B are $64^\circ 15'$ and $48^\circ 20'$, respectively. Find the height of the balloon.

48. If $\cos\phi = n\sin\alpha$, and $\cot\phi = \dfrac{\sin\alpha}{\tan\beta}$, prove $\cos\beta = \dfrac{n}{\sqrt{1 + n^2\cos^2\alpha}}$.

49. Find $\cos 3\alpha$, when $\tan 2\alpha = -\frac{4}{3}$.

50. Solve completely the triangle, given $a = 0.296$, $B = 28^\circ 47'.3$, $C = 84^\circ 25'$.

51. Evaluate $\sin 300^\circ + \cos 240^\circ + \tan 225^\circ$.

52. Evaluate $\sec\dfrac{2\pi}{3} - \csc\dfrac{5\pi}{3} + \tan\dfrac{4\pi}{3}$.

53. If
$$\tan\theta = \frac{\sin\alpha\cos\gamma - \sin\beta\sin\gamma}{\cos\alpha\cos\gamma - \cos\beta\sin\gamma}$$
and
$$\tan\phi = \frac{\sin\alpha\sin\gamma - \sin\beta\cos\gamma}{\cos\alpha\sin\gamma - \cos\beta\cos\gamma},$$
show that $\tan(\theta + \phi) = \tan(\alpha + \beta)$.

54. If $\tan 466^\circ\, 15'\, 38'' = -\frac{24}{7}$, find the sine and cosine of $233^\circ\, 7'\, 49''$.

55. Prove $\dfrac{\csc\alpha - \cot\alpha}{\sec\alpha + \tan\alpha} = \dfrac{\sec\alpha - \tan\alpha}{\csc\alpha + \cot\alpha}$.

56. Prove $\dfrac{\cos(\alpha - 3\beta) - \cos(3\alpha - \beta)}{\sin 2\alpha + \sin 2\beta} = 2\sin(\alpha - \beta)$.

57. Prove $\sin 80^\circ = \sin 40^\circ + \sin 20^\circ$.

58. Prove $\cos 20^\circ = \cos 40^\circ + \cos 80^\circ$.

59. Prove $4\tan^{-1}\dfrac{1}{5} - \tan^{-1}\dfrac{1}{239} = \dfrac{\pi}{4}$.

60. From the deck of a ship a rock bears N.N.W. After the ship has sailed 10 miles E.N.E., the rock bears due W. Find its distance from the ship at each observation.

61. Find the length of an arc of 80° in a circle of 4 feet radius.

62. Given $\tan\theta = \frac{4}{3}$, $\tan\phi = \frac{5}{12}$, evaluate $\sin(\theta + \phi) + \cos(\theta - \phi)$.

63. If $\tan\theta = 2\tan\phi$, show that $\sin(\theta + \phi) = 3\sin(\theta - \phi)$.

64. Prove $\cos(\alpha+\beta)\cos(\alpha-\beta) + \sin(\alpha+\beta)\sin(\alpha-\beta) = \dfrac{1-\tan^2\beta}{1+\tan^2\beta}$.

65. Solve $4\cos 2\theta + 3\cos\theta = 1$.

66. Solve $3\sin\alpha = 2\sin(60^\circ - \alpha)$.

67. Prove $(\sin\alpha - \csc\alpha)^2 - (\tan\alpha - \cot\alpha)^2 + (\cos\alpha - \sec\alpha)^2 = 1$.

68. Prove $2(\sin^6\alpha + \cos^6\alpha) + 1 = 3(\sin^4\alpha + \cos^4\alpha)$.

69. Prove $\csc 2\beta + \cot 4\beta = \cot\beta - \csc 4\beta$.

70. If $\tan p = \dfrac{5}{12}$, $\cos 2q = \dfrac{527}{625}$, then $\csc\dfrac{p-q}{2} = 5\sqrt{13}$.

71. Solve completely the triangle, given
$$a = 0.0654,\quad b = 0.092,\quad B = 38^\circ\, 40'.4.$$

72. Solve completely the triangle, given
$$b = 10,\quad c = 26,\quad B = 22^\circ\, 37'.$$

73. A railway train is travelling along a curve of $\frac{1}{3}$ mile radius at the rate of 25 miles per hour. Through what angle (in circular measure) will it turn in half a minute?

74. Express the following angles in circular measure:

$$63^\circ, \quad 4^\circ\,30', \quad 6^\circ\,12'\,36''.$$

75. Express the following angles in sexagesimal measure:

$$\frac{\pi}{6}, \quad \frac{3\pi}{8}, \quad \frac{17\pi}{64}.$$

76. If A, B, C are angles of a triangle, prove

$$\cos A + \cos B + \cos C = 1 + 4 \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

77. Prove $\sin 2x + \sin 2y + \sin 2z = 4 \sin x \sin y \sin z$, when x, y, z are the angles of a triangle.

78. Prove $\sec \alpha = 1 + \tan \alpha \tan \frac{\alpha}{2}$.

79. Prove $\sin^2(\alpha + \beta) - \sin^2(\alpha - \beta) = \sin 2\alpha \sin 2\beta$.

80. Prove $\cos^2(\alpha + \beta) - \sin^2(\alpha - \beta) = \cos 2\alpha \cos 2\beta$.

81. Prove $\dfrac{\sin 19p + \sin 17p}{\sin 10p + \sin 8p} = 2 \cos 9p$.

82. Consider with reference to their ambiguity the triangles whose known parts are:

(*a*)	$a = 2743$,	$b = 6452$,	$B = 43^\circ\,15'$;
(*b*)	$a = 0.3854$,	$c = 0.2942$,	$C = 38^\circ\,20'$;
(*c*)	$b = 5$,	$c = 53$,	$B = 15^\circ\,22'$;
(*d*)	$a = 20$,	$b = 90$,	$A = 63^\circ\,28'.5$.

83. From a ship at sea a lighthouse is observed to bear S.E. After the ship sailed N.E. 6 miles the bearing of the lighthouse is S. 27° 30′ E. Find the distance of the lighthouse at each time of observation.

84. Prove $\dfrac{\sin(\theta + 3\phi) + \sin(3\theta + \phi)}{\sin 2\theta + \sin 2\phi} = 2\cos(\theta + \phi)$.

85. Prove $\cos 15^\circ - \sin 15^\circ = \dfrac{1}{\sqrt{2}}$.

86. Show that $\cos(\alpha + \beta)\cos(\alpha - \beta) = \cos^2\alpha - \sin^2\beta$
$$= \cos^2\beta - \sin^2\alpha.$$

87. Show that $\tan(\alpha + 45^\circ)\tan(\alpha - 45^\circ) = \dfrac{2\sin^2\alpha - 1}{2\cos^2\alpha - 1}$.

88. Solve $\sin(x + y)\sin(x - y) = \frac{1}{2}$, $\cos(x + y)\cos(x - y) = 0$.

89. Prove $\dfrac{1 + \sin\alpha - \cos\alpha}{1 + \sin\alpha + \cos\alpha} = \tan\dfrac{\alpha}{2}$.

90. Prove $\tan 2\theta + \sec 2\theta = \dfrac{\cos\theta + \sin\theta}{\cos\theta - \sin\theta}$.

91. If $\tan\phi = \dfrac{b}{a}$, then $a\cos 2\phi + b\sin 2\phi = a$.

92. Prove $\sin^{-1}\dfrac{1}{\sqrt{5}} + \cot^{-1} 3 = \dfrac{\pi}{4}$.

93. Solve $\cos A + \cos 7A = \cos 4A$.

94. Two sides of a triangle, including an acute angle, are 5 and 7 the area is 14; find the other side.

95. Show that $\dfrac{3\cos 3\theta - 2\cos\theta - \cos 5\theta}{\sin 5\theta - 3\sin 3\theta + 4\sin\theta} = \tan 2\theta$.

96. A regular pyramid stands on a square base one side of which is 173.6 feet. This side makes an angle of 67° with one edge. What is the height of the pyramid?

97. From points directly opposite on the banks of a river 500 yards wide the mast of a ship lying between them is observed to be at an elevation of 10° 28′.4 and 12° 14′.5, respectively. Find the height of the mast.

98. Show that $(\sin 60° - \sin 45°)(\cos 30° + \cos 45°) = \sin^2 30°$.

99. Find x if $\sin^{-1} x + \sin^{-1}\dfrac{x}{2} = \dfrac{\pi}{4}$.

100. Trace the changes in sign and value of $\sin\alpha + \cos\alpha$ as α changes from 0° to 360°.

CHAPTER VIII.

MISCELLANEOUS PROPOSITIONS.

71. The circle inscribed in a given triangle is often called the *incircle* of the triangle, its centre the *incentre*, and its radius is denoted by r. The incentre is the point of intersection of the three bisectors of the angles of the triangle (geometry).

The circle circumscribed about a triangle is called the *circumcircle*, its centre the *circumcentre*, and its radius R. The circumcentre is the point of intersection of perpendiculars erected at the middle points of the three sides of the triangle (geometry).

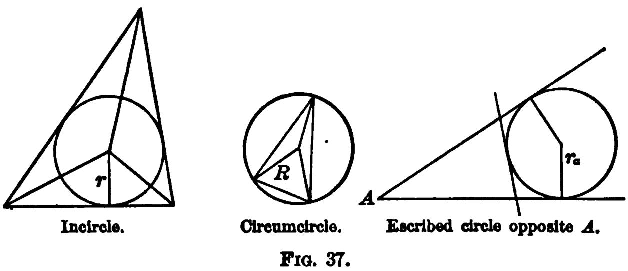

FIG. 37.

The circle which touches any side of a triangle and the other two sides produced is called the *escribed* circle; its radius is denoted by r_a, r_b, or r_c, according as the escribed circle is opposite angle A, B, or C.

Again, the altitudes from the vertices of a triangle meet in a point called the *orthocentre* of the triangle.

Finally, the medians of a triangle meet in a point called the *centroid*, which is two-thirds of the length of the median from the vertex of the angle from which that median is drawn (geometry).

Certain properties of the above will now be considered.

72. To find the radius of the incircle.

Let Δ, Δ', Δ'', Δ''' represent the areas of triangles ABC, COB, AOC, BOA, respectively.

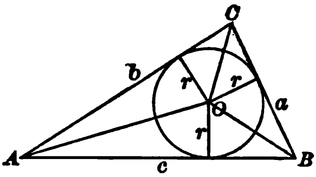

FIG. 38.

Then

$$\Delta = \Delta' + \Delta'' + \Delta'''$$
$$= \tfrac{1}{2}(a + b + c)r = sr.$$

And since $\quad \Delta = \sqrt{s(s-a)(s-b)(s-c)}$, (Art. 63)

$$\therefore\ r = \frac{\Delta}{s} = \frac{\sqrt{(s-a)(s-b)(s-c)}}{s}.$$

COR. To express the angles in terms of r and the sides, divide each member of the above equation by $s - a$.

Then $\quad \dfrac{r}{s-a} = \sqrt{\dfrac{(s-b)(s-c)}{s(s-a)}} = \tan \tfrac{1}{2} A.$ (Art. 62)

In like manner $\tan \frac{1}{2} B = \dfrac{r}{s-b}$; $\tan \frac{1}{2} C = \dfrac{r}{s-c}$.

73. To find the radius of the circumcircle.

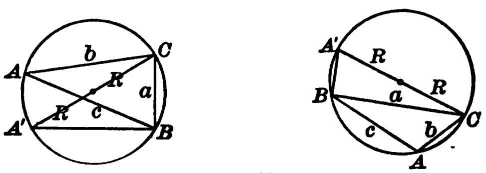

FIG. 39.

In the figure ABC is the given triangle, and $A'C$ a diameter of the circumcircle. Then, angle $A = A'$, or $180° - A'$.

$$\therefore\ \sin A = \sin A'.$$

Since $A'BC$ is a right angle,

$$\sin A' = \frac{BC}{A'C} = \frac{a}{2R}.$$

$$\therefore\ R = \frac{\frac{1}{2}a}{\sin A}.$$

COR. 1. As above, $2R=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$, which is another proof of the "law of sines."

COR. 2. From $R=\frac{a}{2\sin A}$, we have

$$R=\frac{abc}{2bc\sin A}=\frac{abc}{4\Delta}, \text{ where } \Delta=\text{area } ABC.$$

74. To find the radii of the escribed circles.

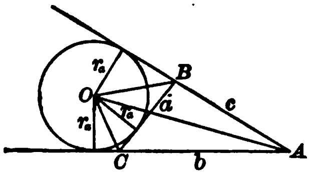

FIG. 40.

Represent areas ABC, BOA, AOC, BOC, by Δ, Δ', Δ'', Δ''', respectively. Then r_a is the altitude of each of the triangles BOA, AOC, BOC.

Now
$$\begin{aligned}\Delta &= \Delta'+\Delta''-\Delta''' \\ &= \tfrac{1}{2}r_ac+\tfrac{1}{2}r_ab-\tfrac{1}{2}r_aa \\ &= \tfrac{1}{2}r_a(c+b-a)=r_a(s-a).\end{aligned}$$

$$\therefore\ r_a=\frac{\Delta}{s-a}.$$

In like manner, $r_b=\frac{\Delta}{s-b}$; $r_c=\frac{\Delta}{s-c}$.

75. The orthocentre.

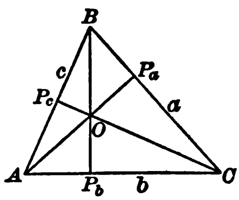

FIG. 41.

Denote the perpendiculars on the sides a, b, c, by AP_a, BP_b, CP_c, and let it be required to find the distances from their intersection O to the sides of the triangle, and also to the vertices.

$$OP_b=AP_b\tan CAO.$$

But $AP_b=c\cos A$, and $CAO=90°-C$.

$$\therefore\ OP_b=c\cos A\cot C=\frac{c}{\sin C}\cos A\cos C.$$

$$=2R\cos A\cos C. \qquad \text{(Art. 73, Cor. 1)}$$

In like manner, $OP_c = 2R \cos B \cos A,$

$$OP_a = 2R \cos C \cos B.$$

Again, the distances from the orthocentre to the vertices are,

$$OA = \frac{AP_b}{\cos CAO} = \frac{c \cos A}{\sin C}$$

$$= 2R \cos A.$$

Also, $$OB = 2R \cos B,$$

and $$OC = 2R \cos C.$$

76. Centroid and medians.

The lengths of the medians may be computed as follows:

In the figure the medians to the sides a, b, c, are AM_a, BM_b, CM_c, meeting in the centroid O.

Now, by the law of cosines, from the triangle BM_bC,

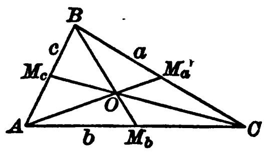

FIG. 42.

$$BM_b^2 = a^2 + M_bC^2 - 2a \cdot M_bC \cdot \cos C$$

$$= a^2 + \frac{b^2}{4} - ab \cos C.$$

But, $\cos C = \dfrac{a^2 + b^2 - c^2}{2ab},$

$$\therefore\ BM_b^2 = a^2 + \frac{b^2}{4} - \frac{a^2 + b^2 - c^2}{2} = \frac{2a^2 + 2c^2 - b^2}{4},$$

whence, $BM_b = \frac{1}{2}\sqrt{2a^2 + 2c^2 - b^2} = \frac{1}{2}\sqrt{a^2 + c^2 + 2ac \cos B}$

since $$\frac{a^2 + c^2 - b^2}{2ac} = \cos B.$$

In like manner,

$$CM_c = \tfrac{1}{2}\sqrt{2b^2 + 2a^2 - c^2} = \tfrac{1}{2}\sqrt{b^2 + a^2 + 2ba \cos C},$$

and $$AM_a = \tfrac{1}{2}\sqrt{2c^2 + 2b^2 - a^2} = \tfrac{1}{2}\sqrt{c^2 + b^2 + 2cb \cos A}.$$

EXAMPLES.

1. In the triangle, $a = 25$, $b = 35$, $c = 45$, find R, r, r_a.

2. Given $a = 0.354$, $b = 0.548$, $C = 28° 34' 20''$, find the distances to C and B, from the circumcentre, the incentre, the centroid, and the orthocentre.

3. In the ambiguous triangle show that the circumcircles of the two triangles, when there are two solutions, are equal.

4. Prove that $\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$.

5. In any triangle prove $\Delta = \sqrt{r\, r_a r_b r_c}$.

6. Prove that the product of the distances of the incentre from the vertices of the triangle is $4\, r^2 R$.

7. Prove that the area of all triangles of given perimeter that can be circumscribed about a given circle is constant.

8. Prove that the area of the triangle ABC is $Rr(\sin A + \sin B + \sin C)$.

CHAPTER IX.

SERIES — DE MOIVRE'S THEOREM — HYPERBOLIC FUNCTIONS.

77. First consider some series by means of which logarithms of numbers and the natural functions of angles may be computed. For this purpose the following series is important:

$$e = 1 + \frac{1}{\underline{|1}} + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \cdots.$$

It may be derived as follows:

By the binomial theorem,

$$\left(1 + \frac{1}{n}\right)^{nx}$$

$$= 1 + nx \cdot \frac{1}{n} + \frac{nx(nx-1)}{\underline{|2}} \cdot \frac{1}{n^2} + \frac{nx(nx-1)(nx-2)}{\underline{|3}} \cdot \frac{1}{n^3} + \cdots$$

$$= 1 + x + \frac{x\left(x - \frac{1}{n}\right)}{\underline{|2}} + \frac{x\left(x - \frac{1}{n}\right)\left(x - \frac{2}{n}\right)}{\underline{|3}} + \cdots,$$

and if n increase without limit,

$$= 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots + \frac{x^r}{\underline{|r}} + \cdots$$

This is called the *exponential series*, and is represented by e^x, so that

$$e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots + \frac{x^r}{\underline{|r}} + \cdots.$$

It is shown in higher algebra that this equation holds for all values of x; whence, if $x = 1$,

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \cdots + \frac{1}{\underline{|r}} + \cdots.$$

This value of e is taken as the base of the natural or Naperian system of logarithms.

This value e, however, is not the base of the system of logarithms computed by Napier, but its reciprocal instead. The natural logarithm is used in the theoretical treatment of logarithms, and, as will presently appear, it is customary to compute the common logarithm by first finding the natural, and then multiplying it by a constant multiplier called the modulus, Art. 82; *i.e.* in the Naperian system the modulus is taken as 1, and the base is computed. In the common system the base 10 is chosen and the modulus computed.

78. From the exponential series the value of e may be computed to any required degree of accuracy.

$$e = 1 + 1 + \frac{1}{\underline{|2}} + \frac{1}{\underline{|3}} + \frac{1}{\underline{|4}} + \cdots.$$

$$1 + 1 + \frac{1}{\underline{|2}} = 2.5$$

$$\frac{1}{\underline{|3}} = 0.1666666666$$

$$\frac{1}{\underline{|4}} = 0.0416666666$$

$$\frac{1}{\underline{|5}} = 0.0083333333$$

$$\frac{1}{\underline{|6}} = 0.0013888888$$

$$\frac{1}{\underline{|7}} = 0.0001984126$$

$$\frac{1}{\underline{|8}} = 0.0000248015$$

$$\frac{1}{\underline{|9}} = 0.0000027557$$

$$\frac{1}{\underline{|10}} = 0.0000002755$$

. . . .

Adding, $e = 2.7182818$, correct to 7 decimal places.

79. To expand a^x in ascending powers of x.

$$e^z = 1 + z + \frac{z^2}{\lfloor 2} + \frac{z^3}{\lfloor 3} + \cdots + \frac{z^r}{\lfloor r} + \cdots.$$

Let $a^x = e^z$, then $z = \log_e a^x = x \cdot \log_e a$. (Arts. 35, 40)

Substituting

$$a^x = 1 + x \cdot \log_e a + \frac{x^2(\log_e a)^2}{\lfloor 2} + \frac{x^3(\log_e a)^3}{\lfloor 3} + \cdots.$$

Now put $1 + a$ for a, and

$$(1+a)^x = 1 + x \cdot \log_e(1+a) + \frac{x^2[\log_e(1+a)]^2}{\lfloor 2} + \frac{x^3[\log_e(1+a)]^3}{\lfloor 3} + \cdots.$$

But by the binomial theorem,

$$(1+a)^x = 1 + xa + \frac{x(x-1)}{\lfloor 2} a^2 + \frac{x(x-1)(x-2)}{\lfloor 3} a^3 + \cdots.$$

Equating coefficients of x in the second members of the above equations,

$$\log_e(1+a) = a - \frac{a^2}{2} + \frac{a^3}{3} - \frac{a^4}{4} + \cdots;$$

or writing x for a,

$$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

In this form the series is of little practical use, since it converges very slowly, and only when x is between $+1$ and -1 (higher algebra).

Put $-x$ for x, and

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots;$$

$$\therefore \log_e(1+x) - \log_e(1-x)$$

$$= \log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

Finally, put $\dfrac{1}{2n+1}$ for x, and

$$\log_e \frac{n+1}{n} = \log_e(n+1) - \log_e n$$

$$= 2\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \cdots\right\},$$

$$\therefore \log_e(n+1) = \log_e n + 2\left\{\frac{1}{2n+1} + \frac{1}{3}\left(\frac{1}{2n+1}\right)^3 + \frac{1}{5}\left(\frac{1}{2n+1}\right)^5 + \cdots\right\},$$

a series which is rapidly convergent.

80. From this series a table of logarithms to the base e may be computed.

To find $\log_e 2$ put $n = 1$. Then, since $\log_e 1 = 0$, the series becomes

$$\log_e 2 = \log_e 1 + 2\left\{\frac{1}{3} + \frac{1}{3 \cdot 3^3} + \frac{1}{5 \cdot 3^5} + \frac{1}{7 \cdot 3^7} + \frac{1}{9 \cdot 3^9}\right.$$
$$\left. + \frac{1}{11 \cdot 3^{11}} + \frac{1}{13 \cdot 3^{13}} + \cdots\right\} = 0.693147.$$

The computations may be arranged thus:

3	2.00000000	
9	.66666667	= .66666667
9	.07407407 ÷ 3	= .02469136
9	.00823045 ÷ 5	= .00164609
9	.00091449 ÷ 7	= .00013064
9	.00010161 ÷ 9	= .00001129
9	.00001129 ÷ 11	= .00000103
	.00000125 ÷ 13	= .00000009
		.69314717

whence $\log_e 2 = 0.693147$, correct to 6 decimal places.

To find $\log_e 3$, put $n = 2$, and

$$\log_e 3 = \log_e 2 + 2\left(\frac{1}{5} + \frac{1}{3 \cdot 5^3} + \frac{1}{5 \cdot 5^5} + \frac{1}{7 \cdot 5^7} + \frac{1}{9 \cdot 5^9} + \cdots\right).$$

$$\begin{array}{r|lcl}
5 & 2.00000000 & & \\
25 & .40000000 & = & .40000000 \\
25 & .01600000 \div 3 = & & .00533333 \\
25 & .00640000 \div 5 = & & .00012800 \\
25 & .00025600 \div 7 = & & .00000366 \\
 & .00000102 \div 9 = & & \underline{.00000011} \\
 & & & .40546510 \\
 & & \log_e 2 = & \underline{.69314717} \\
 & & \therefore \log_e 3 = & 1.098612,
\end{array}$$

correct to 6 decimal places.

$\log_e 4 = 2 \times \log_e 2$, $\log_e 6 = \log_e 3 + \log_e 2$, etc. (Why?) The logarithms of prime numbers may be computed as above by giving proper values to n.

81. Having computed the logarithms of numbers to base e, the logarithms to any other base may be computed by means of the following relation:

Let $\log_a n = x$; then $a^x = n$.

Also, $\log_b n = y$; then $b^y = n$,

$$\therefore a^x = b^y.$$

Hence, $\log_a(a^x) = \log_a(b^y)$,

and $\therefore x = y \log_a b$.

It follows that $\log_a n = \log_b n \cdot \log_a b$;

whence $$\log_b n = \log_a n \cdot \frac{1}{\log_a b}.$$

This factor $\frac{1}{\log_a b}$ is called the *modulus* of the system of logarithms to base b. Using it as a multiplier, logarithms of numbers to base b are computed at once from the logarithms of the same numbers to any other base a.

82. To compute the common logarithms.

Common logarithms are computed from the Naperian by use of the modulus $\frac{1}{\log_e 10}$; *i.e.*

$$\log_{10} n = \log_e n \cdot \frac{1}{\log_e 10}.$$

By Art. 80, $\log_e 10$ can be found, and

$$\frac{1}{\log_e 10} = .434294, \textit{ the modulus of the common system.}$$

Ex. Compute the common logarithms of:

2, 3, 4, 6, 5, 10, 15, 216, 3375.

COMPLEX NUMBERS.

83. In algebra it is shown that the general expression for complex numbers is $a + bi$, where a represents all the real terms of the expression, b the coefficients of all the imaginary terms, and i is so defined that $i^2 = -1$; whence

$$i = \sqrt{-1}, \quad i^2 = -1, \quad i^3 = -i, \quad i^4 = 1, \text{ etc.}$$

The laws of operation in algebra are found to apply to complex numbers. Moreover, it is further shown that if two complex numbers are equal, the real terms are equal, and the imaginary terms are equal; *i.e.* if

$$a + bi = c + di,$$

then $$a = b \text{ and } c = d.$$

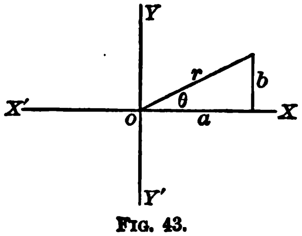

FIG. 43.

Finally, the complex number may be graphically represented as follows:

The real number is measured along OX, a units; the imaginary parallel to OY, b units. The line r is a graphic representation of $a + bi$.

Since $\qquad a = r\cos\theta$ and $b = r\sin\theta$,

$$\therefore\ a + bi = r(\cos\theta + i\sin\theta).$$

The properties of complex numbers are best developed by using this trigonometric form. If r be taken as unity, then $\cos\theta + i\sin\theta$ represents any complex number.

84. De Moivre's Theorem. To prove that, for any value of n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

I. When n is a positive integer.

By multiplication,

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)$$
$$= \cos\alpha\cos\beta - \sin\alpha\sin\beta + i(\sin\alpha\cos\beta + \cos\alpha\sin\beta)$$
$$= \cos(\alpha+\beta) + i\sin(\alpha+\beta).$$

In like manner,

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)$$
$$= \cos(\alpha+\beta+\gamma) + i\sin(\alpha+\beta+\gamma);$$

and finally,

$$(\cos\alpha + i\sin\alpha)(\cos\beta + i\sin\beta)(\cos\gamma + i\sin\gamma)\cdots \text{ to } n \text{ factors}$$
$$= \cos(\alpha+\beta+\gamma+\cdots) + i\sin(\alpha+\beta+\gamma+\cdots).$$

Now let $\alpha = \beta = \gamma = \cdots$, and the above becomes

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha.$$

II. When n is a negative integer.

Let $n = -m$; then

$$(\cos\alpha + i\sin\alpha)^n = (\cos\alpha + i\sin\alpha)^{-m}$$
$$= \frac{1}{(\cos\alpha + i\sin\alpha)^m} = \frac{1}{\cos m\alpha + i\sin m\alpha}$$
$$= \frac{\cos m\alpha - i\sin m\alpha}{(\cos m\alpha + i\sin m\alpha)(\cos m\alpha - i\sin m\alpha)}$$
$$= \frac{\cos m\alpha - i\sin m\alpha}{\cos^2 m\alpha + \sin^2 m\alpha}$$
$$= \cos m\alpha - i\sin m\alpha = \cos(-m)\alpha + i\sin(-m)\alpha.$$

Substituting n for $-m$, the equation becomes

$$(\cos\alpha + i\sin\alpha)^n = \cos n\alpha + i\sin n\alpha.$$

III. When n is a fraction, positive or negative.

Let $n = \frac{p}{q}$, p and q being any integers.

Now

$$\left(\cos\frac{\alpha}{q} + i\sin\frac{\alpha}{q}\right)^q = \cos q\cdot\frac{\alpha}{q} + i\sin q\cdot\frac{\alpha}{q} = \cos\alpha + i\sin\alpha \quad \text{(by I)}.$$

Then $$\left(\cos\frac{\alpha}{q} + i\sin\frac{\alpha}{q}\right) = (\cos\alpha + i\sin\alpha)^{\frac{1}{q}}.$$

Raising each member to the power p,

$$(\cos\alpha + i\sin\alpha)^{\frac{p}{q}} = \left(\cos\frac{\alpha}{q} + i\sin\frac{\alpha}{q}\right)^p = \cos\frac{p}{q}\alpha + i\sin\frac{p}{q}\alpha.$$

COMPUTATIONS OF NATURAL FUNCTIONS.

85. The radian measure of an acute angle is greater than its sine and less than its tangent, *i.e.*

$$\sin\alpha < \alpha < \tan\alpha.$$

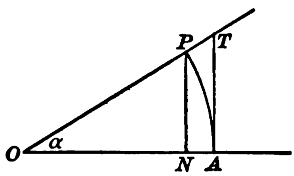

FIG. 44.

Let α be the circular, or radian, measure of any acute angle AOP.

Then, in the figure,

area of sector $OAP <$ area of triangle OAT,

i.e. $\frac{1}{2}OA\cdot\text{arc}\,AP < \frac{1}{2}OA\cdot AT.$

$\therefore\ \text{arc}\,AP < AT.$

Now, since $NP < \text{arc}\,AP$,

$$\frac{NP}{OP} < \frac{\text{arc}\,AP}{OP} < \frac{AT}{OP}.$$

But $\frac{\text{arc}\,AP}{OP} =$ circular measure of $AOP = \alpha$;

whence $\sin\alpha < \alpha < \tan\alpha.$

86. Since $$\sin\alpha < \alpha < \tan\alpha,$$

$$1 < \frac{\alpha}{\sin\alpha} < \frac{1}{\cos\alpha}.$$

Hence, however small α may be, $\frac{\alpha}{\sin\alpha}$ lies between 1 and $\frac{1}{\cos\alpha}$. When α approaches 0, $\cos\alpha$ approaches unity. Therefore, by diminishing α sufficiently, we may make $\frac{\alpha}{\sin\alpha}$ differ from unity by an amount less than any assignable quantity. This we express by saying that when α approaches 0, $\frac{\alpha}{\sin\alpha}$ approaches unity as a limit, *i.e.* $\frac{\alpha}{\sin\alpha}=1$, approximately. Multiplying by $\cos\alpha$ ($=1$, nearly), we have $\frac{\alpha}{\tan\alpha}=1$, approximately. Whence, if α approaches 0, $\tan\alpha=\sin\alpha=\alpha$, approximately.

87. Sine and cosine series.

$\cos n\alpha + i\sin n\alpha = (\cos\alpha + i\sin\alpha)^n$, (De Moivre's Theorem).

Expanding the second member by the binomial formula, it becomes,

$$\cos^n\alpha + n\cos^{n-1}\alpha\cdot i\sin\alpha + \frac{n(n-1)}{\underline{|2}}\cos^{n-2}\alpha\,.\,i^2\sin^2\alpha$$
$$+\frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\alpha\cdot i^3\sin^3\alpha$$
$$+\frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\alpha\cdot i^4\sin^4\alpha+\cdots.$$

Substituting the values of i^2, i^3, i^4, etc., we have

$$\cos n\alpha + i\sin n\alpha = \cos^n\alpha - \frac{n(n-1)}{\underline{|2}}\cos^{n-2}\alpha\sin^2\alpha$$
$$+\frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\alpha\sin^4\alpha - \cdots$$
$$+ i\left(n\cos^{n-1}\alpha\sin\alpha - \frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\alpha\sin^3\alpha + \cdots\right).$$

Equating the real and imaginary parts in the two members,

$$\cos n\alpha = \cos^n \alpha - \frac{n(n-1)}{\underline{|2}} \cos^{n-2}\alpha \sin^2 \alpha + \frac{n(n-1)(n-2)(n-3)}{\underline{|4}} \cos^{n-4}\alpha \sin^4\alpha - \cdots,$$

and $$\sin n\alpha = n \cos^{n-1}\alpha - \frac{n(n-1)(n-2)}{\underline{|3}} \cos^{n-3}\alpha \sin^3 \alpha + \cdots.$$

Ex. 1. Find $\cos 3\alpha$; $\sin 3\alpha$.

In the above put $n = 3$, and
$$\cos 3\alpha = \cos^3\alpha - 3\cos\alpha \sin^2\alpha$$
$$= 4\cos^3\alpha - 3\cos\alpha;$$
also sin 3α
$$\sin^3 \alpha = 3\cos^2\alpha \sin\alpha - \sin^3\alpha$$
$$= 3\sin\alpha - 4\sin^3\alpha.$$

2. Find $\sin 4\alpha$; $\cos 4\alpha$; $\sin 5\alpha$; $\cos 5\alpha$.

It will be noticed that in the series for $\cos n\alpha$ and $\sin n\alpha$ the terms are alternately positive and negative, and that the series continues till there is a zero factor in the numerator.

88. If now in the above series we let $n\alpha = \theta$, then

$$\cos\theta = \cos^n\alpha - \frac{\frac{\theta}{\alpha}\left(\frac{\theta}{\alpha} - 1\right)}{\underline{|2}} \cos^{n-2}\alpha \sin^2\alpha + \frac{\frac{\theta}{\alpha}\left(\frac{\theta}{\alpha}-1\right)\left(\frac{\theta}{\alpha}-2\right)\left(\frac{\theta}{\alpha}-3\right)}{\underline{|4}} \cos^{n-4}\alpha \sin^4\alpha - \cdots$$

$$= \cos^n\alpha - \frac{\theta(\theta-\alpha)}{\underline{|2}} \cos^{n-2}\alpha \left(\frac{\sin\alpha}{\alpha}\right)^2 + \frac{\theta(\theta-\alpha)(\theta-2\alpha)(\theta-3\alpha)}{\underline{|4}} \cos^{n-4}\alpha \left(\frac{\sin\alpha}{\alpha}\right)^4 - \cdots.$$

If now θ remain constant, and α decrease without limit, then will n become indefinitely great, and $\frac{\sin\alpha}{\alpha}$ and every

power thereof, and $\cos\alpha$ and every power of $\cos\alpha$ will approach unity as a limit, so that

$$\cos\theta = 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \cdots.$$

Similarly, $$\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \cdots.$$

By algebra it is shown that these series are convergent for all values of θ. By their use we can compute values of *sin* θ and *cos* θ to any required degree of accuracy.

Show from the above that $\tan\theta = \theta + \frac{\theta^3}{3} + \frac{2\,\theta^5}{15} + \cdots.$

Ex. 1. Compute the value of $\sin 1°$, correct to 5 places.

In $\sin\theta = \theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \cdots$, make θ the radian measure of $1° = \frac{\pi}{180} = 0.01745 +.$

Then, $$\theta = 0.01745 +$$

$$\frac{\theta^3}{\underline{|3}} = 0.0000008.$$

$$\therefore \sin\theta = 0.01745 +.$$

The terms of the series after the first do not affect the fifth place, so that the value is given by the first term, an illustration of the fact that, if α is small, $\sin\alpha = \alpha$, approximately. Compare the value of $\tan 1°$.

2. Show that $\sin 10° = 0.17365$; $\cos 10° = 0.98481$; $\sin 15° = 0.25882$; $\cos 60° = 0.50000$.

3. Find the sine and cosine of $18° 30'$; $22° 15'$; $67° 45'$.

It is unnecessary to compute the functions beyond 30°, for since

$$\sin(30° + \theta) + \sin(30° - \theta) = \cos\theta \text{ (why?)},$$

$$\therefore \sin(30° + \theta) = \cos\theta - \sin(30° - \theta).$$

So, also, $$\cos(30° + \theta) = \cos(30° - \theta) - \sin\theta.$$

Giving θ proper values the functions of any angle from 30° to 45° are determined at once from the functions of angles less than 30°.

Thus, $$\sin 31° = \cos 1° - \sin 29°;$$

$$\cos 31° = \cos 29° - \sin 1°.$$

4. Find sine and cosine of 40°; of 50°.

89. The following are sometimes useful in applied mathematics:

Ex. 1. To find the sum of a series of sines of angles in A. P., such as

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+[n-1]\beta).$$

$$2\sin\alpha\sin\frac{\beta}{2} = \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{\beta}{2}\right),$$

$$2\sin(\alpha+\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{3\beta}{2}\right),$$

$$2\sin(\alpha+2\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{3\beta}{2}\right) - \cos\left(\alpha+\frac{5\beta}{2}\right),$$

$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$

$$2\sin(\alpha+[n-1]\beta)\sin\frac{\beta}{2} = \cos\left(\alpha+\frac{2n-3}{2}\beta\right) - \cos\left(\alpha+\frac{2n-1}{2}\beta\right).$$

Adding

$$2\{\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+[n-1]\beta)\}\sin\frac{\beta}{2}$$

$$= \cos\left(\alpha-\frac{\beta}{2}\right) - \cos\left(\alpha+\frac{2n-1}{2}\beta\right)$$

$$= 2\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n}{2}\beta.$$

$$\therefore \sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \cdots + \sin(\alpha+[n-1]\beta)$$

$$= \frac{\sin\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n}{2}\beta}{\sin\frac{\beta}{2}}.$$

Similarly it can be shown that

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \cdots + \cos(\alpha+[n-1]\beta)$$

$$= \frac{\cos\left(\alpha+\frac{n-1}{2}\beta\right)\sin\frac{n}{2}\beta}{\sin\frac{\beta}{2}}.$$

90. The series $e^x = 1 + x + \frac{x^2}{\underline{|2}} + \frac{x^3}{\underline{|3}} + \cdots + \frac{x^r}{\underline{|r}} + \cdots$ is proved in higher algebra to be true for all values of x, real or imaginary. Then if $x = i\theta$,

$$e^{i\theta} = 1 + i\theta + \frac{i^2\theta^2}{\underline{|2}} + \frac{i^3\theta^3}{\underline{|3}} + \cdots + \frac{i^r\theta^r}{\underline{|r}} + \cdots$$

$$= 1 - \frac{\theta^2}{\underline{|2}} + \frac{\theta^4}{\underline{|4}} - \frac{\theta^6}{\underline{|6}} + \cdots + i(\theta - \frac{\theta^3}{\underline{|3}} + \frac{\theta^5}{\underline{|5}} - \frac{\theta^7}{\underline{|7}} + \cdots).$$

$$\therefore\ e^{i\theta} = \cos\theta + i\sin\theta \text{ (Art. 87).}$$

In like manner, $e^{-i\theta} = \cos\theta - i\sin\theta.$

Adding, $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2};$

subtracting, $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}.$

HYPERBOLIC FUNCTIONS.

91. Since $\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$, and $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}$ are true for all values of θ, let $\theta = i\theta$.

Then, $\sin(i\theta) = \frac{e^{-\theta} - e^{\theta}}{2i} = i\frac{e^{\theta} - e^{-\theta}}{2} = i\sinh\theta,$

and $\cos(i\theta) = \frac{e^{\theta} + e^{-\theta}}{2} = \cosh\theta,$

so that $\tan(i\theta) = \frac{\sin(i\theta)}{\cos(i\theta)} = \frac{i\sinh\theta}{\cosh\theta} = i\tanh\theta,$

where $\sinh\theta$, $\cosh\theta$, $\tanh\theta$, are called the *hyperbolic sine, cosine*, and *tangent* of θ. The hyperbolic cotangent, secant, and cosecant of θ are obtained from the hyperbolic sine, cosine, and tangent, just as the corresponding circular functions, cotangent, secant, and cosecant, are obtained from tangent, cosine, and sine. The hyperbolic functions have the same geometric relations to the rectangular hyper-

bola that the circular functions have to the circle, hence the name hyperbolic functions.

$$\sinh\theta = \frac{e^\theta - e^{-\theta}}{2}, \quad \therefore \operatorname{csch}\theta = \frac{2}{e^\theta - e^{-\theta}};$$

$$\cosh\theta = \frac{e^\theta + e^{-\theta}}{2}, \quad \therefore \operatorname{sech}\theta = \frac{2}{e^\theta + e^{-\theta}};$$

$$\tanh\theta = \frac{e^\theta - e^{-\theta}}{e^\theta + e^{-\theta}}, \quad \therefore \coth\theta = \frac{e^\theta + e^{-\theta}}{e^\theta - e^{-\theta}}.$$

92. From the relations of Art. 91 it appears that to any relation between the circular functions there corresponds a relation between the hyperbolic functions.

Since $$\cos^2(i\theta) + \sin^2(i\theta) = 1,$$

$$\cosh^2\theta + i^2\sinh^2\theta = 1,$$

or $$\cosh^2\theta - \sinh^2\theta = 1.$$

This may also be derived thus:

$$\cosh^2\theta - \sinh^2\theta = \left(\frac{e^\theta + e^{-\theta}}{2}\right)^2 - \left(\frac{e^\theta - e^{-\theta}}{2}\right)^2$$

$$= \frac{e^{2\theta} + 2 + e^{-2\theta} - e^{2\theta} + 2 - e^{-2\theta}}{4} = 1.$$

Also since

$$\sin(i\alpha + i\beta) = \sin(i\alpha)\cos(i\beta) + \cos(i\alpha)\sin(i\beta),$$

$$\therefore i\sinh(\alpha+\beta) = i\sinh\alpha\cosh\beta + \cosh\alpha \cdot i\sinh\beta,$$

and $$\sinh(\alpha+\beta) = \sinh\alpha\cosh\beta + \cosh\alpha\sinh\beta.$$

Let the student verify this relation from the exponential values of sinh and cosh.

EXAMPLES.

Prove

1. $\cosh(\alpha+\beta) = \cosh\alpha\cosh\beta + \sinh\alpha\sinh\beta.$
2. $\cosh(\alpha+\beta) - \cosh(\alpha-\beta) = 2\sinh\alpha\sinh\beta.$
3. $\cosh 2\theta = 1 + 2\sinh^2\theta = 2\cosh^2\theta - 1.$
4. $\sinh 2\alpha = 2\sinh\alpha\cosh\alpha.$

5. $\cosh\frac{\theta}{2}=\sqrt{\frac{1+\cosh\theta}{2}}$; $\sinh\frac{\theta}{2}=\sqrt{\frac{\cosh\theta-1}{2}}$.

6. $\sinh 3\theta = 3\sinh\theta + 4\sinh^3\theta$.

7. $\sinh\theta + \sinh\phi = 2\sinh\frac{\theta+\phi}{2}\cosh\frac{\theta-\phi}{2}$.

8. $\sinh\alpha+\sinh(\alpha+\beta)+\sinh(\alpha+2\beta)+\cdots+\sinh(\alpha+[n-1]\beta)$

$$=\frac{\sinh\left(\alpha+\frac{n-1}{2}\beta\right)\sinh\frac{n}{2}\beta}{\sinh\frac{\beta}{2}}$$

9. $\tanh(\theta+\phi)=\frac{\tanh\theta+\tanh\phi}{1+\tanh\theta\tanh\phi}$.

10. $\sinh^{-1}x = \cosh^{-1}\sqrt{1+x^2} = \tanh^{-1}\frac{x}{\sqrt{1+x^2}}$.

11. $\cosh(\alpha+\beta)\cosh(\alpha-\beta) = \cosh^2\alpha + \sinh^2\beta = \cosh^2\beta + \sinh^2\alpha$.

12. $2\cosh n\alpha\cosh\alpha = \cosh(n+1)\alpha + \cosh(n-1)\alpha$.

13. $\cosh\alpha = \frac{1}{2}(e^{\alpha}+e^{-\alpha}) = 1+\frac{\alpha^2}{\lfloor 2}+\frac{\alpha^4}{\lfloor 4}+\cdots$.

14. $\sinh\alpha = \frac{1}{2}(e^{\alpha}-e^{-\alpha}) = \alpha+\frac{\alpha^3}{\lfloor 3}+\frac{\alpha^5}{\lfloor 5}+\cdots$.

15. $\tanh^{-1}a + \tanh^{-1}b = \tanh^{-1}\frac{a+b}{1+ab}$.

SPHERICAL TRIGONOMETRY.

CHAPTER X.

SPHERICAL TRIANGLES.

93. Spherical trigonometry is concerned chiefly with the solution of spherical triangles. Its applications are for the most part in geodesy and astronomy.

The following definitions and theorems of geometry are for convenience of reference stated here.

A *great circle* is a plane section of a sphere passing through the centre. Other plane sections are *small circles*.

The shortest distance between two points on a sphere is measured on the arc of a great circle, less than 180°, which joins them.

A *spherical triangle* is any portion of the surface of a sphere bounded by three arcs of great circles. We shall consider only triangles whose sides are arcs not greater than 180° in length.

The *polar triangle* of any spherical triangle is the triangle whose sides are drawn with the vertices of the first triangle as poles. If ABC is the polar of $A'B'C'$, then $A'B'C'$ is the polar of ABC.

In any spherical triangle,

The sum of two sides $>$ *the third side.*

The greatest side is opposite the greatest angle, and conversely.

Each angle $< 180°$, *the sum of the angles* $> 180°$, *and* $< 540°$.

Each side $< 180°$; *the sum of the sides* $< 360°$.

The sides of a spherical triangle are the supplements of the angles opposite in the polar triangle, and conversely.

If two angles are equal the sides opposite are equal, and conversely.

The sides of a spherical triangle subtend angles at the centre of the sphere which contain the same number of angle degrees as the arc does of arc degrees; *i.e.* an angle at the centre and its arc have the same measure numerically.

The arc does not measure the angle for they have not the same unit of measurement, but we say they have the same numerical measure; *i.e.* the arc contains the unit arc as many times as the angle contains the unit angle.

The angles of a spherical triangle are said to be measured by the plane angle included by tangents to the sides of the angle at their intersection. They have therefore the same numerical measure as the dihedral angle between the planes of the arcs.

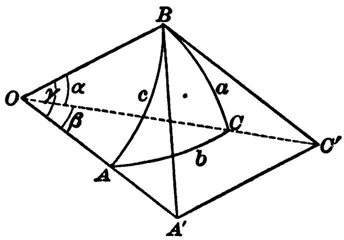

FIG. 45.

In the figure the following have the same numerical measure:

arc a and angle α;

arc b and angle β;

arc c and angle γ;

plane angle $A'BC'$;

spherical angle B and dihedral angle $A\text{–}BO\text{–}C$;

spherical angle C and dihedral angle $B\text{–}CO\text{–}A$;

spherical angle A and dihedral angle $C\text{–}AO\text{–}B$.

$A'C'B$ and $C'A'B$ have not the same measure as spherical angles C and A, for BA', $A'C'$, $C'B$ are not perpendicular to OA or OC.

94. In plane trigonometry the trigonometric functions were treated as functions of the angles. But since an angle and its subtending arc vary together and have the same

numerical measure, it is clear that the trigonometric ratios are functions of the arcs, and may be so considered. All the relations between the functions are the same whether we consider them with reference to the angle or the arc, so that all the identities of plane trigonometry are true for the functions of the arcs.

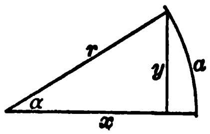

FIG. 46.

Thus in the figure we may write,

$$\sin\alpha = \frac{y}{r} \text{ or } \sin a = \frac{y}{r};$$

$$\sin^2\alpha + \cos^2\alpha = 1, \text{ or } \sin^2 a + \cos^2 a = 1;$$

$$\cos 2\alpha = 2\cos^2\alpha - 1, \text{ or } \cos 2a = 2\cos^2 a - 1.$$

GENERAL FORMULÆ FOR SPHERICAL TRIANGLES.

95. The solutions of spherical triangles may be effected by formulæ now to be developed:

First it will be shown that in any spherical triangle

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos b = \cos c \cos a + \sin c \sin a \cos B,$$

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

The following cases must be considered:

I. Both b and $c < 90°$. III. Both b and $c > 90°$.

II. $b > 90°$, $c < 90°$. IV. Either b or $c = 90°$.

V. $b = c = 90°$.

The figure applies to Case I.

Let ABC be a spherical triangle, a, b, c its sides, and O the centre of the sphere.

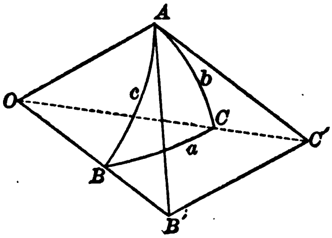

FIG. 47.

Draw AC' and AB' tangent to the sides b, c at A. (The same result would be obtained by drawing AB', AC' perpendicular to OA at any point to

meet OB, OC.) Since these tangents lie in the planes of the circles to which they are drawn, they will meet OC and OB in C' and B', and the angle $C'AB'$ will be the measure of the angle A of the spherical triangle ABC. Since OAB', OAC' are right angles, AOB', AOC' must be acute, and hence sides c, b are each $< 90°$.

In the triangles $C'AB'$ and $C'OB'$,

$$C'B'^2 = AC'^2 + AB'^2 - 2\,AC' \cdot AB' \cos C'AB',$$

and $$B'C'^2 = OC'^2 + OB'^2 - 2\,OC' \cdot OB' \cos C'OB'.$$

Subtracting and noting that

$$\cos C'AB' = \cos A \text{ and } \cos C'OB' = \cos a,$$

we have

$$0 = OC'^2 - AC'^2 + OB'^2 - AB'^2$$
$$+ 2\,AC' \cdot AB' \cos A - 2\,OC' \cdot OB' \cos a.$$

But $OC'^2 - AC'^2 = OA^2$ and $OB'^2 - AB'^2 = OA^2$.

Hence, $$0 = OA^2 + AC' \cdot AB' \cos A - OC' \cdot OB' \cos a;$$

or $$\cos a = \frac{OA}{OC'} \cdot \frac{OA}{OB'} + \frac{AC'}{OC'} \cdot \frac{AB'}{OB'} \cos A.$$

$$\therefore \cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Similarly,

$$\cos b = \cos a \cos c + \sin a \sin c \cos B,$$

and $$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

These formulæ are important, and should be carefully memorized.

II. $b > 90°$; $c < 90°$.

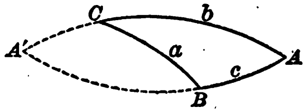

FIG. 48.

In the triangle ABC, let $b > 90°$ and $c < 90°$. Complete the lune $BACA'$. Then in the triangle $A'CB$ the sides a and $A'C$ are both less than 90°, and by (I)

$$\cos A'B = \cos A'C \cos a + \sin A'C \sin a \cos A'CB.$$

But $A'B = 180° - c$, $A'C = 180° - b$, and $A'CB = 180° - C$.

$$\therefore \cos(180° - c) = \cos(180° - b)\cos a + \sin(180° - b)\sin a \cos(180° - C);$$

or $$-\cos c = (-\cos b)\cos a + \sin b \sin a(-\cos C),$$

and $$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

A similar proof will apply in case $c > 90°$, $b < 90°$.

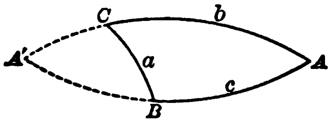

FIG. 49.

III. Both b and $c > 90°$.

In the triangle ABC, let both b and $c > 90°$. Complete the lune $ABA'C$. Then since $A'C$ and $A'B$ are both $< 90°$,

$$\cos a = \cos A'C \cos A'B + \sin A'C \sin A'B \cos A'.$$

But $A' = A$, $A'C = 180° - b$, $A'B = 180° - c$.

$$\therefore \cos a = \cos(180° - b)\cos(180° - c) + \sin(180° - b)\sin(180° - c)\cos A;$$

or $$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

Cases IV and V are left to the student as exercises.

96. Since the angles of the polar triangle are the supplements of the sides opposite in the first triangle, we have

$$a' = 180° - A,\quad b' = 180° - B,$$
$$c' = 180° - C,\quad A' = 180° - a.$$

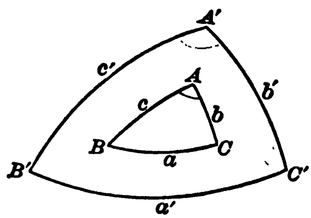

FIG. 50.

Substituting in

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A',$$

we have

$$\cos(180° - A) = \cos(180° - B)\cos(180° - C) + \sin(180° - B)\sin(180° - C)\cos(180° - a);$$

or $$-\cos A = (-\cos B)(-\cos C) + \sin B \sin C(-\cos a).$$

Changing signs,

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a.$$

Similarly, $$\cos B = -\cos A \cos C + \sin A \sin C \cos b,$$

and $$\cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

97. In any spherical triangle to prove $\dfrac{\sin A}{\sin a} = \dfrac{\sin B}{\sin b} = \dfrac{\sin C}{\sin c}$.

Since $$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c};$$

$$\therefore \sin^2 A = 1 - \left(\frac{\cos a - \cos b \cos c}{\sin b \sin c}\right)^2$$

$$= \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{(1 - \cos^2 b)(1 - \cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

Hence,

$$\sin A = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c - 2 \cos a \cos b \cos c}}{\sin b \sin c}$$

and $$\frac{\sin A}{\sin a} = \frac{\sqrt{1 - \cos^2 a - \cos^2 b - \cos^2 c - 2 \cos a \cos b \cos c}}{\sin a \sin b \sin c}.$$

By a similar process, $\dfrac{\sin B}{\sin b}$ and $\dfrac{\sin C}{\sin c}$ will be found equal to the same expression.

$$\therefore \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

98. Expressions for sine, cosine, and tangent of half an angle in terms of functions of the sides.

We have $2\sin^2\frac{A}{2} = 1 - \cos A$

$$= 1 - \frac{\cos a - \cos b\cos c}{\sin b\sin c}$$

$$= \frac{\cos b\cos c + \sin b\sin c - \cos a}{\sin b\sin c}$$

$$= \frac{\cos(b-c) - \cos a}{\sin b\sin c}.$$

Then $2\sin^2\frac{A}{2} = \frac{2\sin\frac{1}{2}(a+b-c)\sin\frac{1}{2}(a-b+c)}{\sin b\sin c}$ (Art. 51)

$$= \frac{2\sin(s-b)\sin(s-c)}{\sin b\sin c},$$

when $2s = a + b + c.$

$$\therefore \sin\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b\sin c}}.$$

Similarly, $\sin\frac{B}{2} = \sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin a\sin c}},$

and $\sin\frac{C}{2} = \sqrt{\frac{\sin(s-b)\sin(s-a)}{\sin a\sin b}}.$

Also from the relation

$$2\cos^2\frac{A}{2} = 1 + \cos A$$

$$= 1 + \frac{\cos a - \cos b\cos c}{\sin b\sin c},$$

we have $\cos\frac{A}{2} = \sqrt{\frac{\sin s\sin(s-a)}{\sin b\sin c}}.$

Also, $\cos\frac{B}{2} = \sqrt{\frac{\sin s\sin(s-b)}{\sin c\sin a}},$

and $\cos\frac{C}{2} = \sqrt{\frac{\sin s\sin(s-c)}{\sin a\sin b}}.$

From the above,

$$\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

Also,
$$\tan\frac{B}{2}=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}},$$

and
$$\tan\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}.$$

Compare the formulæ thus far derived with the corresponding formulæ for solving plane triangles. The similarity in forms will assist in memorizing the formulæ for solving spherical triangles.

99. From the formulæ of Art. 96, the student can easily prove the following relations:

$$\sin\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\sin B\sin C}},$$

where
$$2S=A+B+C.$$

$$\sin\frac{b}{2}=?,$$

$$\sin\frac{c}{2}=?.$$

$$\cos\frac{a}{2}=\sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B\sin C}},$$

$$\cos\frac{b}{2}=?,$$

$$\cos\frac{c}{2}=?.$$

$$\tan\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}},$$

$$\tan\frac{b}{2}=?,$$

$$\tan\frac{c}{2}=?.$$

100. Napier's Analogies.

Since
$$\frac{\tan\frac{A}{2}}{\tan\frac{B}{2}}=\frac{\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}}{\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\sin(s-b)}}}$$
$$=\sqrt{\frac{\sin^2(s-b)}{\sin^2(s-a)}}=\frac{\sin(s-b)}{\sin(s-a)};$$

by composition and division,

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{\tan\frac{A}{2}-\tan\frac{B}{2}}=\frac{\sin(s-b)+\sin(s-a)}{\sin(s-b)-\sin(s-a)},$$

$$\frac{\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}+\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}}{\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}-\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}}=\frac{\sin\frac{1}{2}(2s-a-b)\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(2s-a-b)\sin\frac{1}{2}(a-b)},\quad\text{(Art. 51)}$$

$$\frac{\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}=\frac{\tan\frac{1}{2}(2s-a-b)}{\tan\frac{1}{2}(a-b)}$$

$$=\frac{\tan\frac{c}{2}}{\tan\frac{1}{2}(a-b)},\quad\text{since } 2s-a-b=c.$$

$$\therefore\ \tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

To find an expression for $\tan\frac{1}{2}(A-B)$ we have only to consider the polar triangle, and by substituting $180°-A$ for a, etc., $180°-a$ for A, etc., we have the following relations:

$$\tfrac{1}{2}(a-b)=\tfrac{1}{2}(180°-A-180°+B)=-\tfrac{1}{2}(A-B);$$

also, $\frac{1}{2}(A-B)=-\frac{1}{2}(a-b)$;

$$\tfrac{1}{2}(A+B)=\tfrac{1}{2}(180^\circ-a+180^\circ-b)=180^\circ-\tfrac{1}{2}(a+b);$$

and
$$\frac{c}{2}=90^\circ-\frac{C}{2}.$$

The formula then becomes, applying Art. 29,

$$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

Formulæ for $\tan\frac{1}{2}(a+b)$, $\tan\frac{1}{2}(A+B)$ are derived as follows:

Since

$$\tan\frac{A}{2}\cdot\tan\frac{B}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\cdot\sin(s-a)}}\cdot\sqrt{\frac{\sin(s-c)\sin(s-a)}{\sin s\cdot\sin(s-b)}},$$

$$\frac{\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}}=\frac{\sin(s-c)}{\sin s}.$$

By composition and division,

$$\frac{\cos\frac{A}{2}\cos\frac{B}{2}+\sin\frac{A}{2}\sin\frac{B}{2}}{\cos\frac{A}{2}\cos\frac{B}{2}-\sin\frac{A}{2}\sin\frac{B}{2}}=\frac{\sin s+\sin(s-c)}{\sin s-\sin(s-c)},$$

whence
$$\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}=\frac{\tan\frac{1}{2}(a+b)}{\tan\frac{c}{2}}, \qquad \text{(Art. 51)}$$

since
$$2s-c=a+b,$$

or,
$$\tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

The value of $\tan\frac{1}{2}(A+B)$ is derived by substituting in terms of the corresponding elements of the polar triangle.

$$\frac{\cos\frac{1}{2}(a-b)}{-\cos\frac{1}{2}(a+b)}=\frac{-\tan\frac{1}{2}(A+B)}{\cot\frac{C}{2}},$$

$$\therefore\ \tan\frac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

Similar relations among the other elements of the triangle may be derived, or they may be written from the above by proper changes of A, B, C, a, b, c in the formulæ. The student should write them out as exercises.

101. Delambre's Analogies.

Since $$\sin\frac{1}{2}(A+B)=\sin\frac{A}{2}\cos\frac{B}{2}+\cos\frac{A}{2}\sin\frac{B}{2},$$

then

$$\sin\frac{1}{2}(A+B)=\frac{\sin(s-b)+\sin(s-a)}{\sin c}\cdot\sqrt{\frac{\sin s\cdot\sin(s-c)}{\sin a\cdot\sin b}}.$$

(Art. 98)

Hence, $$\frac{\sin\frac{1}{2}(A+B)}{\cos\frac{C}{2}}=\frac{\sin(s-b)+\sin(s-a)}{\sin c}$$

$$=\frac{2\sin\frac{c}{2}\cos\frac{1}{2}(a-b)}{2\sin\frac{c}{2}\cos\frac{c}{2}},\quad \text{(Art. 51)}$$

and $$\sin\frac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{c}{2}}\cos\frac{C}{2};$$

In like manner derive

$$\sin\frac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{c}{2}}\cos\frac{C}{2};$$

$$\cos\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a+b)}{\cos\frac{c}{2}}\sin\frac{C}{2};$$

$$\cos\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a+b)}{\sin\frac{c}{2}}\sin\frac{C}{2}.$$

These formulæ are often called Gauss's Formulæ, but they were first discovered by Delambre in 1807. Afterwards Gauss, independently, discovered them, and published them in his *Theoria Motus.*

102. Formulæ for solving *right spherical triangles* are derived from the foregoing by putting $C=90°$, whence $\sin C=1,\ \cos C=0$.

$$\cos c=\cos a\cos b+\sin a\sin b\cos C \qquad \text{(Art. 95)}$$

becomes
$$\cos c=\cos a\cos b. \qquad (1)$$

Substituting the value of $\cos a$ from (1), and simplifying,

$$\cos A=\frac{\cos a-\cos b\cos c}{\sin b\sin c} \qquad \text{(Art. 95)}$$

becomes
$$\cos A=\frac{\tan b}{\tan c}. \qquad (2)$$

Again,
$$\frac{\sin A}{\sin a}=\frac{\sin C}{\sin c} \qquad \text{(Art. 97)}$$

in the right triangle is

$$\sin A=\frac{\sin a}{\sin c}. \qquad (3)$$

Dividing (3) by (2),

$$\tan A=\frac{\sin a\cos b}{\cos c\sin b}=\frac{\sin a\cos a\cos b}{\cos c\cos a\sin b}=\frac{\sin a}{\cos a\sin b},$$

since
$$\cos a\cos b=\cos c.$$

$$\therefore\ \tan A\sin b=\tan a. \qquad (4)$$

From (4) $\tan a = \tan A \sin b,$

also, $\tan b = \tan B \sin a.$

Multiplying, $\tan a \tan b = \tan A \tan B \sin a \sin b,$

or, $$\cot A \cot B = \cos a \cos b = \cos c. \tag{5}$$

From (2) and (3), by division,

$$\frac{\cos A}{\sin B} = \frac{\dfrac{\tan b}{\tan c}}{\dfrac{\sin b}{\sin c}} = \frac{\cos c}{\cos b} = \cos a.$$

$$\therefore \cos A = \cos a \sin B. \tag{6}$$

Let the student write formulæ (2), (3), (4), (6) for B. It will be noticed that (1) and (5) give values for c only, while (2), (3), (4), (6) apply only to A and B.

103. Formulæ (1)–(6) are sufficient for the solution of right spherical triangles if any two parts besides the right angle are given. They are easily remembered by comparison with corresponding formulæ in plane trigonometry. Two rules, invented by Napier, and called *Napier's Rules of Circular Parts*, include all the formulæ of Art. 102.

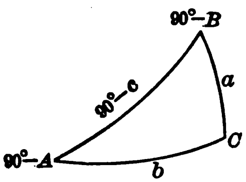

Fig. 51.

Omitting C, and taking the complements of A, c, and B, the parts of the triangle taken in order are a, b, $90° - A$, $90° - c$, $90° - B$. These are called the *circular parts* of the triangle.

Any one of the five parts may be selected as the *middle part*, the two parts next to it are called the *adjacent parts*, and the remaining two the *opposite parts*. Thus, if a be taken as the middle part, $90° - B$ and b are the adjacent parts, and $90° - c$, $90 - A$ the opposite parts.

Napier's Two Rules are as follows:

The sine of the middle part equals the product of the tangents of the adjacent parts.

The sine of the middle part equals the product of the cosines of the opposite parts.

It will aid the memory somewhat to notice that **i** occurs in sine and middle, **a** in tangent and adjacent, and **o** in cosine and opposite, these words being associated in the rules.

The value of the above rules is frequently questioned, most computers preferring to associate the formulæ with the corresponding formulæ of plane trigonometry.

These rules may be proved by taking each of the parts as the middle part, and showing that the formulæ derived from the rules reduce to one of the six formulæ of Art. 102.

Then, if b is the middle part, by the rules,

$$\sin b = \tan a \tan (90^\circ - A) = \tan a \cot A, \text{ or } \tan A = \frac{\tan a}{\sin b},$$

$$\sin b = \cos (90^\circ - c) \cos (90^\circ - B) = \sin c \sin B,$$

or

$$\sin B = \frac{\sin b}{\sin c},$$

results which agree with (4) and (3), Art. 102. If any other part be taken as the middle part, the rules will be found to hold.

104. Area of the spherical triangle.

If r = radius of the sphere,

E = spherical excess of the triangle $= A + B + C - 180^\circ$,

Δ = area of spherical triangle, then by geometry

$$\Delta = Er^2 \times \frac{\pi}{180}.$$

If the three angles are not known, E may be computed by one of the following methods, and Δ found as above.

Cagnoli's Method.

$$\sin\frac{E}{2} = \sin\tfrac{1}{2}(A + B + C - 180^\circ)$$

$$= \sin\tfrac{1}{2}(A + B)\sin\frac{C}{2} - \cos\tfrac{1}{2}(A + B)\cos\frac{C}{2}$$

$$= [\cos\tfrac{1}{2}(a - b) - \cos\tfrac{1}{2}(a + b)]\frac{\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{c}{2}} \quad \text{(Art. 101)}$$

$$= \frac{2\sin\frac{a}{2}\sin\frac{b}{2}}{\cos\frac{c}{2}} \cdot \frac{\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}}{\sin a\sin b}. \quad \text{(Arts. 51, 98)}$$

$$\sin\frac{E}{2} = \frac{\sqrt{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}}{2\cos\frac{a}{2}\cos\frac{b}{2}\cos\frac{c}{2}}.$$

Lhuilier's Method.

$$\tan\frac{E}{4} = \frac{\sin\frac{1}{4}(A + B + C - 180^\circ)}{\cos\frac{1}{4}(A + B + C - 180^\circ)}.$$

Now, multiply each term of the fraction by

$$2\cos\tfrac{1}{4}(A + B - C + 180^\circ),$$

and by Art. 51, (1) and (3), the equation becomes

$$\tan\frac{E}{4} = \frac{\sin\frac{1}{2}(A + B) - \cos\frac{C}{2}}{\cos\frac{1}{2}(A + B) + \sin\frac{C}{2}}$$

$$= \frac{\left[\cos\frac{1}{2}(a - b) - \cos\frac{c}{2}\right]\cos\frac{C}{2}}{\left[\cos\frac{1}{2}(a + b) + \cos\frac{c}{2}\right]\sin\frac{C}{2}} \quad \text{(Art. 101)}$$

$$= \frac{\sin\frac{1}{2}(s - b)\sin\frac{1}{2}(s - a)}{\cos\frac{s}{2}\cos\frac{1}{2}(s - c)} \cdot \sqrt{\frac{\sin s\sin(s - c)}{\sin(s - a)\sin(s - b)}}. \quad \text{(Art. 51)}$$

By Art. 52, introducing the coefficient under the radical,

$$\tan\frac{E}{4}=\sqrt{\tan\frac{s}{2}\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}.$$

If two sides and the included angle are given, E may be determined as follows:

$$\cos\frac{E}{2}=\cos\tfrac{1}{2}(A+B+C-180°)$$

$$=\cos\tfrac{1}{2}(A+B)\sin\frac{C}{2}+\sin\tfrac{1}{2}(A+B)\cos\frac{C}{2}$$

$$=\cos\tfrac{1}{2}(a+b)\sin^2\frac{C}{2}+\cos\tfrac{1}{2}(a-b)\cos^2\frac{C}{2}\quad\text{(Art. 101)}$$

$$=\frac{\cos\frac{a}{2}\cos\frac{b}{2}+\sin\frac{a}{2}\sin\frac{b}{2}\cos C}{\cos\frac{c}{2}}.$$

$$\text{But}\quad \sin\frac{E}{2}=\frac{\sin\frac{a}{2}\sin\frac{b}{2}\cdot 2\sin\frac{C}{2}\cos\frac{C}{2}}{\cos\frac{c}{2}}.\quad\text{(Cagnoli's Method)}$$

Dividing this equation by the above,

$$\tan\frac{E}{2}=\frac{\sin\frac{a}{2}\sin\frac{b}{2}\sin C}{\cos\frac{a}{2}\cos\frac{b}{2}+\sin\frac{a}{2}\sin\frac{b}{2}\cos C}.$$

This formula is not suitable for logarithmic computations. Usually it is better to compute the angles by Napier's Analogies, and solve by $\Delta = Er^2\times\frac{\pi}{180}$.

EXAMPLES.

1. Show that $\cos a = \cos b \cos c + \sin b \sin c \cos A$ becomes $\sec A = 1 + \sec a$, when $a = b = c$.

2. If $a + b + c = \pi$, prove

$$(a)\quad \cos a = \tan \frac{B}{2} \tan \frac{C}{2}.$$

$$(b)\quad \cos^2 \frac{A}{2} = \frac{\cos a}{\sin b \sin c}.$$

$$(c)\quad \sin^2 \frac{A}{2} = \cot b \cot c.$$

$$(d)\quad \cos A + \cos B + \cos C = 1.$$

$$(e)\quad \sin^2 \frac{A}{2} + \sin^2 \frac{B}{2} + \sin^2 \frac{C}{2} = 1.$$

3. Prove $\dfrac{\sin \frac{E}{4} \cos \frac{1}{2}\left(A - \frac{E}{2}\right)}{\sin \frac{A}{2}} = \dfrac{\sin \frac{S}{2} \sin \frac{1}{2}(s - a)}{\cos \frac{a}{2}}.$ (Art. 104)

4. Show that $\cos a \sin b = \sin a \cos b \cos C + \sin c \cos A$.

CHAPTER XI.

SOLUTION OF SPHERICAL TRIANGLES.

105. According to the principles of spherical geometry any three parts are sufficient to *determine* a spherical triangle; the other parts are *computed*, if any three are given, by the formulæ of trigonometry. The known parts may be:

I. Three sides, or three angles.

II. Two sides and the included angle, or two angles and the included side.

III. Two sides and an angle opposite one, or two angles and a side opposite one.

It will appear that, as in plane geometry, III may be ambiguous.

The *signs* of the functions in the formulæ are important since the cosines and tangents of arcs and angles greater than 90° are negative; whether the part sought is greater or less than 90° is therefore determined by the sign of the function in terms of which it is found unless this function be sine. In this case the result is ambiguous, since $\sin\alpha$ and $\sin(180° - \alpha)$ have the same sign and value. Thus if the solution gives $\log\sin a = 9.56504$, we may have either $a = 21° 33'$, or $158° 27'$. The conditions of the problem must determine which values apply to the triangle in question.

The negative signs, when they occur, will be indicated thus:

$$\log\cos 115° 20' = 9.63135^{-},$$

indicating, not that the logarithm is negative, but that in the final result account must be made of the fact that $\cos 115° 20'$ is negative.

106. *Formulæ for the solution of triangles.*

I. $$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}.$$

II. $$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}}.$$

III. $$\tan\frac{a}{2}=\sqrt{\frac{-\cos S\cos(S-A)}{\cos(S-B)\cos(S-C)}}.$$

IV. $$\tan\tfrac{1}{2}(a-b)=\frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

V. $$\tan\tfrac{1}{2}(a+b)=\frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\frac{c}{2}.$$

VI. $$\tan\tfrac{1}{2}(A-B)=\frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

VII. $$\tan\tfrac{1}{2}(A+B)=\frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\frac{C}{2}.$$

VIII. $$\Delta=Er^2\frac{\pi}{180},$$

where E is determined by

$$\tan\frac{E}{4}=\sqrt{\tan\frac{s}{2}\tan\tfrac{1}{2}(s-a)\tan\tfrac{1}{2}(s-b)\tan\tfrac{1}{2}(s-c)}.$$

Right triangles may be solved as special cases of oblique triangles, or by the following:

(1) $\cos c=\cos a\cos b$. (4) $\tan A\sin b=\tan a$.

(2) $\cos A=\dfrac{\tan b}{\tan c}$. (5) $\cot A\cot B=\cos c$.

(3) $\sin A=\dfrac{\sin a}{\sin c}$. (6) $\cos A=\cos a\sin B$.

The formula to be used in any case may be determined by applying Napier's Rule of Circular Parts.

107. In solving a triangle the student should select formulæ

in which all parts save one are known, and solve for that one (see page 77). Referring to Arts. 105 and 106, it will appear that solutions are effected as follows :

Case I by formulæ II, or III, check by I.

Case II by formulæ VI, VII, I, or IV, V, I, check by IV or VI.

Case III by formulæ I, IV, or I, VI, check by VI or IV.

MODEL SOLUTIONS.

108. 1. Given $a = 46° 24'$, $b = 67° 14'$, $c = 81° 12'$. Solve.

$$\tan\frac{A}{2}=\sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s\sin(s-a)}},\quad \tan\frac{B}{2}=\sqrt{\frac{\sin(s-a)\sin(s-c)}{\sin s\sin(s-b)}},$$

$$\tan\frac{C}{2}=\sqrt{\frac{\sin(s-a)\sin(s-b)}{\sin s\sin(s-c)}}.\quad \textit{Check: } \frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}.$$

Arrange and solve as in Example 1, page 80.

Ans. $A = 46° 13'.5$, $B =$, $C =$.

Solve: (1) $A = 96° 45'$, $B = 108° 30'$, $C = 116° 15'$.

(Use formulæ III in the same manner as in Example 1.)

(2) $a = 108° 14'$, $b = 75° 29'$, $c = 56° 37'$.

(3) $A = 57° 50'$, $B = 98° 20'$, $C = 63° 40'$.

2. Given $b = 113° 3'$, $c = 82° 39'$, $A = 138° 50'$. Solve.

$$\tan\tfrac{1}{2}(B+C)=\frac{\cos\frac{1}{2}(b-c)}{\cos\frac{1}{2}(b+c)}\cot\frac{A}{2},\quad \tan\tfrac{1}{2}(B-C)=\frac{\sin\frac{1}{2}(b-c)}{\sin\frac{1}{2}(b+c)}\cot\frac{A}{2},$$

$$\tfrac{1}{2}(B+C)\pm\tfrac{1}{2}(B-C)=B, \text{ or } C,\qquad \sin a=\frac{\sin A\sin b}{\sin B}.$$

$$\textit{Check: } \tan\frac{a}{2}=\frac{\tan\frac{1}{2}(b-c)\sin\frac{1}{2}(B+C)}{\sin\frac{1}{2}(B-C)}.$$

$b = 113°\ 3'$	$\log\cos\frac{1}{2}(b-c) = 9.98453$	$\log\sin\frac{1}{2}(b-c) = 9.41861$
$c = 82° 39'$	$\operatorname{colog}\cos\frac{1}{2}(b+c) = 0.86461^{-}$	$\operatorname{colog}\sin\frac{1}{2}(b+c) = 0.00409$
$\frac{1}{2}(b+c) = 97° 51'$	$\log\cot\frac{A}{2} = 9.57466$	$\log\cot\frac{A}{2} = 9.57466$
$\frac{1}{2}(b-c) = 15° 12'$		
$\frac{1}{2}A = 69° 25'$	$\log\tan\frac{1}{2}(B+C) = 0.42380^{-}$	$\log\tan\frac{1}{2}(B-C) = 8.99736$
	$\frac{1}{2}(B+C) = 110° 39'$	$\frac{1}{2}(B-C) = 5° 40'.6$
	$\frac{1}{2}(B-C) = 5° 40'.6$	
	$\therefore B = 116° 19'.6$	
	and $C = 104° 58'.4$	

	Check:
$\log \sin A = 9.81839$	$\log \tan \frac{1}{2}(b-c) = 9.43408$
$\log \sin b = 9.96387$	$\log \sin \frac{1}{2}(B+C) = 9.97116$
$\text{colog} \sin B = 0.04756$	$\text{colog} \sin \frac{1}{2}(B-C) = 1.00474$
$\log \sin a = 9.82982$	$\log \tan \frac{a}{2} = 0.40998$
$a = 137° 29'$	$a = 137° 29'$

Notice that $\tan \frac{1}{2}(B+C)$ is $-$. Hence, $\frac{1}{2}(B+C)$ is greater than 90°, *i.e.* 110° 39'.

Solve: (1) $A = 68° 40'$, $B = 56° 20'$, $c = 84° 30'$.

(Use formulæ IV, V, I. Compare Example 2.)

(2) $a = 102° 22'$, $b = 78° 17'$, $C = 125° 28'$.

(3) $A = 130° 5'$, $B = 32° 26'$, $c = 51° 6'$.

109. Ambiguous cases. By the principles of geometry the spherical triangle is not necessarily determined by two sides and an angle opposite, nor by two angles and a side opposite. The triangle may be ambiguous. By geometrical principles it is shown that the marks of the ambiguous spherical triangle are:

1. The parts given are two angles and the side opposite one, or two sides and the angle opposite one.

2. The side, or angle, opposite differs from 90° more than the other given side, or angle.

3. Both sides, or angles, given are either greater than 90°, or less than 90°.

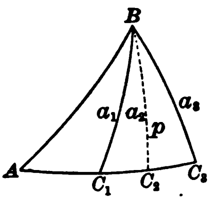

FIG. 52.

In the right triangle ABC_2,

$$\sin a = \sin A \sin c. \quad \text{(formula (3))}$$

Therefore there will be no solution, one solution, or two solutions, according as $\sin a \lesseqgtr \sin A \sin c$, *i.e.* according as $a \lesseqgtr$ the perpendicular p. (See Art. 65.)

But the most expeditious means of determining the ambiguity is found in the solution of the triangle. The use of formula I gives the solution in terms of sine, so that it is to be expected that two values of the part sought may be possible; and whether the triangle be ambiguous or not, there must be some means of determining which of the two

angles, α and $180° - \alpha$, that have the same sine is to be used. If there are two solutions, both values are used.

This is determined in the further solution of the triangle by formula V, which may be written

$$\tan\frac{b}{2} = \frac{\cos\frac{1}{2}(A+C)\tan\frac{1}{2}(a+c)}{\cos\frac{1}{2}(A-C)}.$$

Now $\frac{b}{2} < 90°$, whence $\tan\frac{b}{2}$ is $+$. Then if for both values of C, found by the sine formula, the second member is $+$, there are two solutions; if the second member is $-$ for either value of C, there is but one solution; while if both values of C make the second member $-$, there is no solution. The various cases will be illustrated by problems.

3. Given $a = 62°\,15'.4$, $b = 103°\,18'.8$, $A = 53°\,42'.6$. Solve.

$$\sin B = \frac{\sin b \sin A}{\sin a}, \qquad \tan\frac{c}{2} = \frac{\cos\frac{1}{2}(A+B)\tan\frac{1}{2}(a+b)}{\cos\frac{1}{2}(A-B)},$$

$$\sin C = \frac{\sin c \sin A}{\sin a}. \qquad \textit{Check:}\ \cot\frac{C}{2} = \frac{\tan\frac{1}{2}(A+B)\sin\frac{1}{2}(a+b)}{\sin\frac{1}{2}(a-b)}.$$

Solving the first formula gives

$$\log\sin B = 9.94756,$$

whence

$$B_1 = 62°\,24'.4,$$

$$B_2 = 117°\,35'.6.$$

For each of the values B_1 and B_2,

$$\frac{\cos\frac{1}{2}(A+B)\tan\frac{1}{2}(a+b)}{\cos\frac{1}{2}(A-B)}$$

is $+$ and therefore equal to $\tan\frac{c}{2}$. Hence there are two solutions. Find

$$c = 153°\,\ 9'.6, \text{ or } 70°\,25'.4$$

and

$$C = 155°\,43'.2, \text{ or } 59°\,\ 6'.2$$

4. Given $a = 46°\,45'.5$, $A = 73°\,11'.3$, $B = 61°\,18'.2$. Solve.

$$\sin b = \frac{\sin a \sin B}{\sin A}, \qquad \cot\frac{C}{2} = \frac{\tan\frac{1}{2}(A-B)\cos\frac{1}{2}(a+b)}{\cos\frac{1}{2}(a-b)},$$

$$\sin c = \frac{\sin a \sin C}{\sin A}. \qquad \textit{Check:}\ \tan\frac{c}{2} = \frac{\tan\frac{1}{2}(a-b)\sin\frac{1}{2}(A+B)}{\sin\frac{1}{2}(A-B)}.$$

Solving for b gives $\qquad \log \sin b = 9.82446,$

whence $\qquad b_1 = 41° 52'.5,$

and $\qquad b_2 = 138° \; 7'.5.$

For the value b_1 the fraction

$$\frac{\tan \frac{1}{2}(A+B) \cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)}$$

is $+$, but for b_2 $\cos \frac{1}{2}(a+b)$ is $-$, making the fraction $-$, and hence it can not equal $\cot \frac{C}{2}$, which is $+$. There is then but one solution. Find

$$C = 60° 42'.7, \quad c = 41° 35'.1.$$

5. Given $a = 162° 30'$, $A = 49° 50'$, $B = 57° 52'$. Solve.

Solving gives $\qquad \log \sin b = 9.52274,$

whence $\qquad b_1 = 19° 27'.9,$

$\qquad b_2 = 160° 32'.1.$

For both values, b_1 and b_2, $\cos \frac{1}{2}(a+b)$ is $-$. Therefore,

$$\frac{\tan \frac{1}{2}(A+B) \cos \frac{1}{2}(a+b)}{\cos \frac{1}{2}(a-b)}$$

is $-$ and not equal to $\cot \frac{C}{2}$. Hence the triangle is impossible.

Solve, testing for the number of solutions:

(1) $b = 106° 24'.5$, $c = 40° 20'$, $C = 38° 45'.6$.

(2) $a = 80° 50$, $A = 131° 40'$, $B = 65° 25'$.

(3) $a = 60° 31'.4$, $b = 147° 32'.1$, $B = 143° 50'$.

(4) $a = 55° 30'$, $c = 139° \; 5'$, $A = 43° 25'$.

RIGHT TRIANGLES.

110. Right triangles are a special case of oblique triangles, but are usually solved by formulæ (1) to (6), Art. 106. Students should have no difficulty in applying these.

Computers generally question the utility of Napier's Rules of Circular Parts. For those who prefer the rules a problem will be solved by their use.

6. Given $c = 86° 51'$, $B = 18° 3'.5$, $C = 90°$.

The parts sought are a, b, A, and it is immaterial which is computed first. a and A are adjacent to c and B, while b is the middle part of c and B. Then by Napier's first rule

$$\sin (90° - B) = \tan (90° - c) \tan a;$$

or $$\tan a = \frac{\cos B}{\cot c} = \cos B \tan c,$$

which is formula (2).

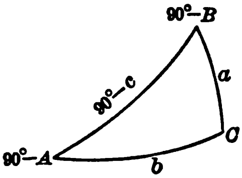

FIG. 53.

By the same rule

$$\sin (90° - c) = \tan (90° - A) \tan (90° - B),$$

or $$\cot A = \frac{\cos c}{\cot B} = \cos c \tan B, \qquad \text{formula (5).}$$

Finally by the second rule

$$\sin b = \cos (90° - c) \cos (90° - B) = \sin c \sin B, \qquad \text{formula (3).}$$

The solutions give $a = 86° 41'.2$, $b = 18° 1'.8$, $A = 88° 58'.4$. Verify.

111. Species. Two angles or sides of a spherical triangle are said to be of the *same species* if they are both less, or both greater, than 90°. They are of *opposite species* when one is greater and the other less than 90°. Since the sides and angles of a spherical triangle may, any or all, be less or greater than 90°, it is necessary in solutions to determine whether each part is more or less than 90°. The directions already given are sufficient in oblique triangles. In right triangles the sign of the function will determine if the solution gives the result in terms of cosine or tangent, but not if the result is found in terms of sine. Thus in Example 6, above, we have $\log \sin b = 9.49068$, whence $b = 18° 1'.8$, or $161° 58'.2$. By formula (4) $\sin b = \frac{\tan a}{\tan A}$. Now $\sin b$ is always +, therefore, $\tan a$ and $\tan A$ must be of the same sign, whence *in any right spherical triangle an oblique angle and its opposite side must be of the same species.*

Again by formula (1) $\cos c = \cos a \cos b$. Now $\cos c$ is + or − according as c is less or greater than 90°. If then $c < 90°$, $\cos a$ and $\cos b$ are of the same sign, but if $c > 90°$, $\cos a$ and $\cos b$ are of opposite sign. Therefore, *if the*

hypotenuse of a right spherical triangle is less than 90°, *the other sides, and hence the angles opposite, are of the same species; but if the hypotenuse be greater than* 90°, *the other sides, and the angles opposite, are of opposite species.*

112. Ambiguous right triangles.

When the parts given are a side adjacent to the right angle, and the angle opposite this side, the triangle is ambiguous, for solving for the hypotenuse by formula (3) gives

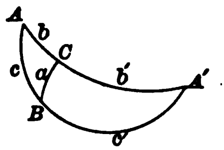

FIG. 54.

$$\sin c = \frac{\sin a}{\sin A},$$

from which there result two values of c. By the last rule of species it follows that to the values of c, one $<90°$, the other $>90°$, there will correspond two values for b, one of the same species as a, the other of opposite species.

Clearly $\sin c \gtreqless 1$, according as $\sin a \gtreqless \sin A$, and hence there will be no solution, one solution, or two solutions, according as $\sin a \gtreqless \sin A$.

Solve the spherical triangles, right angled at C, given:

(1) $b = 73° 21'.4,\quad c = 84° 48'.7.$

(2) $c = 54° 28',\quad B = 128° 12'.6.$

(3) $b = 45° 42',\quad B = 135° 42'.$

(4) $a = 108° 22'.3,\quad b = 120° 14'.5.$

(5) $a = 70° 50',\quad A = 170° 40'.$

(6) $b = 32° 8'.4,\quad B = 46° 2'.8.$

(7) $b = 34° 28',\quad c = 62° 50'.$

(8) $c = 102° 35',\quad B = 17° 45'.$

(9) $a = 92° 16',\quad c = 57° 35'.$

EXAMPLES.

Solve, given:

	a	*b*	*c*	*A*	*B*	*C*
1.	97° 35′	27° 8′.4	119° 8′.4			
2.		67° 33′.4	94° 5′	99° 57′.6		
3.	40° 20′	70° 40′		40°		
4.		82° 39′.5		116° 20′		70° 7
5.	155° 47′.1		110° 46′.4			90°
6.				49° 44′.3	121° 10′.4	26° 6′.3
7.	144° 10′		41° 44′.2		130°	
8.		127° 30′		132° 16′	139° 44′	
9.		155° 5′.3		110° 10′		70° 20′.8
10.	62° 42′			50° 12′	58° 8′	
11.	120° 30′	70° 20′.3	69° 35′			
12.	50° 15′			75° 30′		90°
13.				116° 20′	104° 59′.1	138° 50′.2
14.	84° 14′.5				32° 26′.1	36° 45′.4
15.	100°	50°	60°			
16.			87° 12′	88° 12′		90°
17.	63° 50′	80° 19′		50° 30′		
18.				34° 15′	42° 15′.2	121° 36′.2
19.	50°			63° 15′		90°
20.	159° 50′			159° 43′	123° 40′	
21.	124° 12′.5	54° 18′	97° 12′.5			
22.				48° 31′.3	62° 55′.7	125° 18′.9
23.	76° 36′		40° 20′		42° 15′.2	
24.			28° 45′.1	44° 22′.2	122° 25′.1	
25.		44° 53′	53° 52′			90°
26.	98° 21′.7	109° 50′.4	115° 13′.5			
27.	99° 40′.8	64° 23′.2		95° 38′.1		

APPLICATIONS TO GEODESY AND ASTRONOMY.

113. Geodesy is concerned in measuring portions of the earth's surface, considering the earth as a sphere.

To find the distance on the earth's surface between two points whose latitudes and longitudes are known.

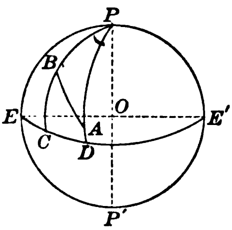

FIG. 55.

If A and B are two places on the earth, P the north pole, $ECDE'$ the equator, and PEP' the principal meridian, *e.g.* the meridian of Greenwich, and if the latitude and longitude of A and B are known, then AB can be computed.

For
$$AP = 90^\circ - \text{latitude } A,$$
$$BP = 90^\circ - \text{latitude } B,$$
$$\text{angle } APB = \text{longitude } A - \text{longitude } B.$$

$\therefore$ two sides and the included angle of the triangle APB are known, and AB can be computed.

Ex. 1. Find the distance between Ann Arbor, 42° 19′ N., 83° 43′.8 W., and San Juan, 18° 29′ N., 66° 7′ W.

2. How far is Manila, 14° 36′ N., 120° 58′ E., from Honolulu, 21° 18′ N., 157° 55′ W.? Honolulu from San Francisco, 37° 47′.9 N., 122° 24′.5 W.? San Francisco from Manila?

114. *The celestial sphere.* The heavenly bodies appear to be situated on a sphere of indefinitely great radius with the centre at the point of observation. This is called the *celestial sphere.*

A tangent plane to the earth at the point of observation cuts the celestial sphere in a great circle called the *horizon.*

The points of the horizon directly south, west, north, east are called the *south, west, north, east points.*

A vertical line through the point of observation cuts the celestial sphere above in the *zenith,* and below in the *nadir,* the zenith and nadir being *poles* of the horizon.

The earth's axis produced is the *axis of the celestial sphere*, cutting it in the north and south poles of the equator.

The *altitude* of a star is its distance from the horizon measured on an arc of a great circle drawn through the star and the zenith.

The *azimuth*, or *bearing*, of a star, is the arc of the horizon measured from some fixed point to the foot of the great circle through the star and the zenith. The fixed point is usually the south point.

The *declination* of a star is its distance from the celestial equator. The circle drawn through the pole and the star is the *hour circle*, and the angle at the pole between the prime meridian and the hour circle is the *hour angle* of the star.

Let an observer be at O on the surface of the earth, and let P be the position of a star.

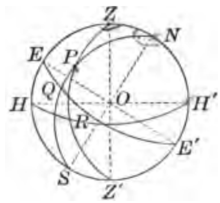

FIG. 56.

Then Z is the zenith, Z' the nadir, EQE' the celestial equator, N its north pole, S its south pole, HRH' the horizon, NPS the meridian, or hour circle, of P, and ZNP the hour angle. The declination of the star is PQ, its altitude PR, and its azimuth, or bearing, NZP. The *astronomical triangle* NZP can be solved if any three of its parts are known.

EXAMPLES.

1. What will be the altitude of the sun at 9 A.M. in Detroit, lat. 42° 20′ N., its declination being 17° 30′.5?

2. At what time will the sun rise at San Francisco, lat. 37° 47′.9, if its declination is 12° 46′.2?

3. Find the azimuth and altitude of a star to an observer in lat. 42° 20′ N., when the hour angle of the star is 3 h. 42.3 m. E., and the declination is 42° 31′ N.

4. The latitude of Sayre Observatory is 40° 36′.4 N.; the sun's altitude is 47° 15′.3, its azimuth 80° 23′.1. Find its declination and hour angle.

5. At Ann Arbor, March 13, 1891, the altitude of Regulus is 32° 10′.3, and the azimuth is 283° 5′.1. Find the declination and hour angle.

$$s = \frac{y}{2\sigma}$$

$$y = 2\sigma g$$